The Gamma Function

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1 The Gamma function

The Gamma function is defined by:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

Notice that:

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-t} t^{x} dt \\
&= \lim_{B\to\infty} \int_0^B e^{-t} t^{x} dt \\
&= \lim_{B\to\infty} \Big[-e^{-t} t^{x} \Big|_0^B + x \int_0^B e^{-t} t^{x-1} dt \Big] && \text{integrating by parts.} \\
&= x \lim_{B\to\infty} \int_0^B e^{-t} t^{x-1} dt \\
&= x\Gamma(x)
\end{aligned}$$

From this we obtain:

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) \\
&= \int_0^\infty e^{-t} dt = 1 \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1 \\
\Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 \\
&\vdots \\
\Gamma(x) &= (x-1)!
\end{aligned}$$

Therefore, the Gamma function is the extension of te factorial, such that, $\Gamma(n+1) = n!$ $\forall n \in \mathbb{Z}$.

1.1 Brief history

Leonhard Euler

Historically, the idea of extending the factorial to non-integers was considered by Daniel Bernoulli and Christian Goldbach in the 1720s. It was solved by Leonhard Euler at the end of the same decade. Euler discovered many interesting properties, such as its reflexion formula: $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$.

James Stirling, contemporary of Euler, also tried to extend the factorial and came up with the Stirling formula, which gives a good approximation of $n!$ but it is not exact. Later on, Carl Gauss, the prince of mathematics, introduced the Gamma function for complex numbers using the Pochhammer factorial. In the early 1810s, it was Adrien Legendre who first used the Γ symbol and named the Gamma function.

1.2 Convergence of the Gamma function

Theorem 1

For every $x > 0$, the following integral converges.

$$\int_0^\infty e^{-t} t^{x-1} dt$$

Proof:

In order to prove this theorem, first we need to show the following lemmas.

Lemma 1.1

For every $x > 0$, the following improper integral converges.

$$\int_0^\infty e^{-xt} dt$$

Proof:

$$\begin{aligned}
\int_0^\infty e^{-xt} dt &= \lim_{B\to\infty} \int_0^B e^{-xt} dt \\
&= \lim_{B\to\infty} \left[\frac{-e^{xt}}{x}\right]\Bigg|_0^B \\
&= \lim_{B\to\infty} \frac{-e^{-Bx} - 1}{x} \\
&= \frac{1}{x}\left[\cancelto{0}{\lim_{B\to\infty} -e^{-Bx}} - \cancelto{1}{\lim_{B\to\infty} 1}\right] \\
&= \frac{1}{x}
\end{aligned}$$

Therefore, the limit exists, then the improper integral converges.

Lemma 1.2
Let n be a natural number. Then,

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = 0$$

Proof:
Using L'Hôpital,

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{(n-1)t^{n-2}}{\frac{1}{2}e^{\frac{1}{2}t}}$$

Since t^{n-1} is a polynomial of degree $n-1$, then

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} t^{n-1} = 0$$

Then, we find by induction that

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{0}{(\frac{1}{2})^n e^{\frac{1}{2}t}} = 0$$

Let's recall the definition of convergence.

$\lim_{n\to\infty} a_n = L$
$a_n \to L \Leftrightarrow \forall \epsilon > 0, \exists N > 0 : \forall n > N, |a_n - L| < \epsilon$

Let $\epsilon = 1$. Then it exists an $M > 0$ such that for every $t \geq M$ the following is true:

$$\left| \frac{t^{n-1}}{e^{\frac{1}{2}t}} \right| < \epsilon = 1$$

Therefore, for every $t \geq M$, $0 \leq t^{n-1} \leq e^{\frac{1}{2}t}$.
This implies that

$$\begin{aligned} 0 &\leq e^{-t} \cdot t^{n-1} \leq e^{-t} \cdot e^{\frac{1}{2}t} \\ 0 &\leq e^{-t} \cdot t^{n-1} \leq e^{-\frac{1}{2}t} \end{aligned} \tag{1}$$

By lemma 1.1, we have that $\int_0^\infty e^{-\frac{1}{2}t} dt$ converges.

(Theorem) Let's recall the comparison test for improper integrals.

Let $f(x)$ and $g(x)$ be two continuous functions on the interval $[\alpha, \infty)$ such that $0 \leq f(x) \leq g(x)$ for every $x \geq \alpha$.
Then, if $\int_{\alpha}^{\infty} g(x)dx$ is convergent, then $\int_{\alpha}^{\infty} f(x)dx$ is too.

Let $f(x) = e^{-t}t^{n-1}$ and $g(x) = e^{-\frac{1}{2}t}$.

Then by the comparison test and (1) we find that

$$\int_0^{\infty} e^{-t}t^{n-1}dt \qquad (2)\ (a)$$

is convergent for every $n \in \mathbb{N}$.

Now, let $x \geq 1$ be any real number.

The floor function $\lfloor x \rfloor$ represents the biggest integer such that $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + 1$.

Then, for $t \geq 0$, we have that

$$0 \leq e^{-t} \cdot t^{x-1} \leq e^{-t} \cdot t^{\lfloor x \rfloor} \qquad (3)$$

By (2) we obtain that the following integral is convergent.

$$\int_0^{\infty} e^{-t}t^{\lfloor x \rfloor}dt$$

Then by the comparison test and (3), we find that

$$\int_0^{\infty} e^{-t}t^{x-1}dt \qquad (b)$$

is convergent for any $x \in \mathbb{R}, x \geq 1$.

To conclude this demonstration, let us study the case when $0 < x < 1$.

We know that

$$\frac{1}{e^{\frac{1}{2}t}} \leq \frac{t^{x-1}}{e^{\frac{1}{2}t}} \leq \frac{t}{e^{\frac{1}{2}t}}$$

By lemma 1.2, we have that

$$\lim_{t\to\infty} \frac{t}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{1}{e^{\frac{1}{2}t}} = 0$$

Using the squeeze theorem,

$$\frac{1}{e^{\frac{1}{2}t}} \leq \frac{t^{x-1}}{e^{\frac{1}{2}t}} \leq \frac{t}{e^{\frac{1}{2}t}}$$
$$0 \leq \frac{t^{x-1}}{e^{\frac{1}{2}t}} \leq 0$$
$$\Rightarrow \frac{t^{x-1}}{e^{\frac{1}{2}t}} = 0$$

for $0 < x < 1$.

In an analogous way, as we did in (1), this implies that

$$0 \leq e^{-t} \cdot t^{x-1} \leq e^{-\frac{1}{2}t}$$

By the comparison test, then

$$\int_0^\infty e^{-t} \cdot t^{x-1} dt \tag{c}$$

is convergent on the interval $0 < x < 1$.

Thus, by (a), (b), and (c), the Gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} \cdot t^{x-1} dt$$

converges for every $x > 0$.

1.3 First and second derivative of $\Gamma(x)$

Differentiating, we find that on the interval $0 < x < \infty$

$$\Gamma'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty e^{-t} t^{x-1} dt = \int_0^\infty e^{-t} t^{x-1} \ln(t) dt$$

$$\Gamma''(x) = \frac{\mathrm{d}}{\mathrm{d}x} \Gamma'(x) = \int_0^\infty e^{-t} t^{x-1} (\ln(t))^2 dt$$

Since the integrand of Γ"(x) is positive for $0 < x < \infty$, then so it is $\Gamma''(x)$. Therefore, the graph of $\Gamma(x)$ is concave up on the interval $(0, \infty)$.

1.4 Extension of the domain of the Gamma function

It is possible to extend the domain of $\Gamma(x)$ to negative values of x.

Let's recall that:

$$\Gamma(x+1) = x\Gamma(x) \tag{4}$$
$$\Rightarrow \Gamma(x) = \frac{\Gamma(x+1)}{x}$$
$$\Rightarrow \Gamma(0) = \frac{\Gamma(1)}{0} \qquad \text{tends to infnity}$$

Using (4) many times, we can see that $\Gamma(-1), \Gamma(-2), \Gamma(-3), \ldots$ also tend to infinity.

Examples:

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = \frac{\Gamma(1)}{-1 \cdot 0}$$

$$\Gamma(-2) = \frac{\Gamma(-1)}{-2} = \frac{\Gamma(0)}{-1 \cdot -2} = \frac{\Gamma(1)}{2 \cdot 0}$$

For any other negative value of x, we can compute $\Gamma(x)$ using (4) until $\Gamma(x+1)$ has a positive argument.

Examples:

$$\Gamma\Big(-\frac{3}{2}\Big) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{-\frac{3}{2} \cdot -\frac{1}{2}} = \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\Big(-\frac{5}{2}\Big) = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \frac{\Gamma(-\frac{1}{2})}{-\frac{5}{2} \cdot -\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{-\frac{5}{2} \cdot -\frac{3}{2} \cdot -\frac{1}{2}} = -\frac{8}{15}\sqrt{\pi}$$

Hence, $\Gamma(x)$ is well defined for any $x \in \mathbb{R}$ except $x = 0, -1, -2, -3, ...$

It is also possible to extend the Gamma function to the complex plane. From the explicit formula, we know it is well defined in the right half plane $\{z \in \mathbb{C} | Re(z) > 0\}$. Then it is holomorphic in the right half plane.

Let us fix z and n such that $Re(z+n) > 0$. Notice that in a neighborhood of $z+n$, the Γ-function is holomorphic.

$$\Gamma(z+n) = (z+n-1)\Gamma(z+n-1) = (z+n-1)(z+n-2)\ldots(z+1)(z)\Gamma(z)$$

Therefore,

$$\Gamma(z) = \frac{\Gamma(z+n)}{P_n(z)}$$

where $P_n(z)$ is the Pochhammer factorial. Since P_n has a root z iff $z \in \{0, -1, -2, -3, ...\}$, then $\Gamma(z)$ is holomorphic iff $P_n(z) \neq 0$, i.e., Γ is holomorphic outside the negative integers.

Hence, $\Gamma(z)$ is a meromorphic function and has poles $z \in \{0, -1, -2, -3, ..\}$.

Now,

$$\frac{1}{\Gamma(x)} = \frac{P_n(z)}{\Gamma(z+n)}$$

Since the gamma function is meromorphic and nonzero everywhere in the complex plane, then its reciprocal is an entire function.

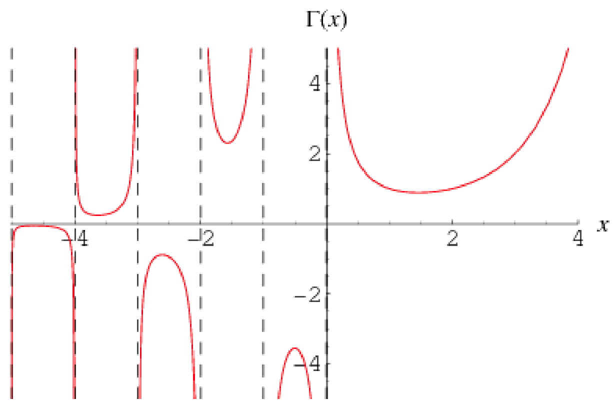

Figure 1: *Gamma Function*

1.5 Incomplete functions of Gamma

The incomplete functions of Gamma are defined by,

$$\gamma(x, \alpha) = \int_0^{\alpha} e^{-t} t^{x-1} dt \qquad \alpha > 0$$

$$\Gamma(x, \alpha) = \int_{\alpha}^{\infty} e^{-t} t^{x-1} dt$$

where it is evident that,

$$\gamma(x, \alpha) + \Gamma(x, \alpha) = \Gamma(x)$$

2 The Beta function

The Beta function is defined by:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

for $x, y > 0$.

Fun fact!

⋆ The Gamma function is also known as the Euler integral of the second kind.
⋆ The Beta function is also known as the Euler integral of the first kind.

2.1 Relationship with Gamma function

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{5}$$

From (5) it is evident that:

$$\beta(x,y) = \beta(y,x)$$

Exercise

Compute $\int\limits_1^3 (x-1)^{10}(x-3)^3 dx$.

Let us make the following change of variables $t = \dfrac{x-1}{2}$

$$\Rightarrow x = 2t+1 \Rightarrow dx = 2t$$

Substituting,

$$\int_0^1 (2t)^{10}(2t-2)^3 2dt = -2^{14}\int_0^1 t^{10}(1-t)^3 dt =$$

$$= -2^{14}\beta(11,4) = -2^{14}\frac{\Gamma(11)\Gamma(4)}{\Gamma(15)} = -2^{14}\cdot\frac{10!\cdot 3!}{14!}$$

Question

Could this method be applied to a family of functions?

2.2 The beta function as an extension of the binomial coefficient

Just like the Gamma function being an extension of the factorial, the Beta function is the extension of the binomial coefficient.

Theorem 2

$$\binom{n}{k} = \frac{1}{(n+1)\beta(n-k+1, k+1)}$$

where n, k in $\mathbb{N}$.

Proof:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n+1) \cdot n!}{(n+1) \cdot k! \cdot (n-k)!} = \frac{(n+1)!}{(n+1) \cdot k!(n-k)} =$$

$$= \frac{1}{n+1} \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)} = \frac{1}{n+1} \frac{1}{\beta(n-k+1, k+1)}$$

Example

$$\binom{3}{2} = \frac{1}{4\beta(2,3)} = \frac{\Gamma(5)}{4\Gamma(2)\Gamma(3)} = 3$$

But also,

$$\binom{3}{2} = \frac{3!}{1! \cdot 2!} = 3$$

Lemma 2.1

$$\beta(n, n+1) = \frac{1}{n \cdot \binom{2n}{n}}$$

Proof:

$$\frac{1}{\binom{2n}{n}} = \frac{1}{\frac{(2n)!}{n! \cdot n!}} = \frac{n! \cdot n!}{(2n)!} =$$

$$= \frac{n \cdot \Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} = n\beta(n, n+1)$$

$$\therefore \beta(n, n+1) = \frac{1}{n \cdot \binom{2n}{n}}$$

Exercise

Find

$$S = \sum_{n=1}^{\infty} \frac{1}{n \cdot \binom{2n}{n}}$$

Solution:

$$S = \sum_{n=1}^{\infty} \frac{1}{n \cdot \binom{2n}{n}}$$

$$= \sum_{n=1}^{\infty} \beta(n+1, n)$$

$$= \sum_{n=1}^{\infty} \int_0^1 t^n (1-t)^{n-1} dt$$

Given the absolute convergence of the integrand, we can switch $\sum$ and $\int$.

$$= \int_0^1 \sum_{n=1}^{\infty} t^n (1-t)^{n-1} dt$$

Using the sum of geometric progressions, we get

$$= \int_0^1 \frac{t}{t^2 - t + 1} dt = \frac{\sqrt{3}}{9}\pi$$

2.3 Incomplete function of Beta

The incomplete function of Beta is defined by,

$$\beta_{\alpha}(x,y) = \int_{0}^{\infty} t^{x-1}(1-t)^{y-1}dt \qquad\qquad 0 \leq \alpha \leq 1$$

2.4 The extension of the permutation

$$nPr = \frac{n!}{(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)}$$

3 Pochhammer factorial

Also known as the rising factorial,

$$x^{(n)} = x(x+1)(x+2)\dots(x+n-1)$$

The rising factorial counts the disposition of things.

Example

$$2^{(2)} = 2 \cdot 3 = 6$$

$$\left\{ \begin{smallmatrix}2\\1\end{smallmatrix},\ 1\ 2,\ \begin{smallmatrix}1\\2\end{smallmatrix},\ \begin{smallmatrix}1\\2\end{smallmatrix},\ 2\ 1,\ \begin{smallmatrix}2\\1\end{smallmatrix} \right\}$$

Example

$$\begin{aligned} 2^{(3)} &= 2 \cdot 3 \cdot 4 = 24 \\ x^{(3)} &= x(x+1)(x+2) = x^3 + 3x^2 + 2x \\ &= 2^3 + 3(2)^2 + 2(2) = 24 \end{aligned}$$

Question

Could this be extended to solve any kind of polynomials?

3.1 Relationship with the Gamma function

$$x^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}$$

$$= \frac{(x+n-1)!}{(x-1)!} = x^{(n)}$$

Definition

Double fatorial

$$n!! = \begin{cases} n(n-2)\ldots 5\cdot 3\cdot 1 & n>0 \text{ odd} \\ n(n-2)\ldots 6\cdot 4\cdot 2 & n>0 \text{ even} \end{cases}$$

Example

$$4!! = 4\cdot 2 = 8$$
$$5!! = 5\cdot 3\cdot 1 = 15$$

Identities

$$n! = n!!\cdot(n-1)!!$$
$$1^{(n)} = n!$$

Example

$$4!! = 4\cdot 2 = 8$$
$$5!! = 5\cdot 3\cdot 1 = 15$$

Exercise

Show that $(2n-1)!! = 2^n \cdot \frac{1}{2}^{(n)}$

Notice that $\frac{1}{2}^{(n)} = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(n-\frac{1}{2}\right)$

Now, observe that if n is impar of the form $2n-1$,

$$\begin{aligned}(2n-1)!! &= (2n-1)(2n-3)\ldots 5\cdot 3\cdot 1 \\ &= 2^n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \\ &= 2^n\left(\frac{1}{2}\right)^{(n)}\end{aligned}$$

$$\Rightarrow (2n-1)!! = 2^n\cdot\frac{1}{2}^{(n)}$$

Exercise

Show that $(2n)!! = 2^n(1)^{(n)}$

Left as an exercise for the reader.

4 Falling factorial

$$x^{(n)} = x(x+1)(x+2)\dots(x+n-1)$$

The falling factorial counts the words of longitude n in order and without repetition.

Example

$$5_{(3)} = 5 \cdot 4 \cdot 3 = 60$$

4.1 Relationship with the Gamma function

$$\begin{aligned} x_{(n)} &= \frac{\Gamma(x+1)}{\Gamma(x-n+1)} \\ &= \frac{x!}{(x-n)!} = x_{(n)} \end{aligned}$$

Fun fact!

The coefficients that appear in the expansion of the falling factorial are the Stirling numbers of the first kind.
Example: $x_{(3)} = x(x-1)(x-2) = x^3 - 3x^2 + 2x$

5 Stirling formula

In this section we are going to study the behavior of the Gamma function for huge positive values of x. It is a good approximation for big factorials.

As n tends to infinity, we have that

$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$

That is,

$$\lim_{n\to\infty} \frac{n}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

Proof:

Using the Gamma function,

$$\Gamma(n+1) = \int_0^\infty e^{-t} t^n dt = n!$$

Making the substitution $t = nk$, we get:

$$= \int_0^\infty e^{-nk} (nk)^n n dk$$
$$= (n)^{n+1} = \int_0^\infty e^{-nk} k^n dk$$

Once again, we substitute s by $(k-1)\sqrt{n}$, we get:

$$= n^{n+1} \int_{-\sqrt{n}}^\infty e^{-n\left(\frac{s}{\sqrt{n}}+1\right)} \left(\frac{s}{\sqrt{n}}+1\right)^n \frac{1}{\sqrt{n}} ds$$
$$= n^n \sqrt{n} e^{-n} \int_{-\sqrt{n}}^\infty e^{-s\sqrt{n}} e^{n \log\left(1+\frac{s}{\sqrt{n}}\right)} ds$$

Let us now consider the Taylor series of $\log(1+r)$,

$$\log(1+r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4} + \dots$$

Making the substitution $r = \frac{s}{\sqrt{n}}$, we get

$$\log\left(1+\frac{s}{\sqrt{n}}\right) = \frac{s}{\sqrt{n}} - \frac{s^2}{2n} + \frac{s^3}{3n\sqrt{n}} - \frac{s^4}{4n^2} + \dots$$

$$n\log\left(1+\frac{s}{\sqrt{n}}\right) = n\left(\frac{s}{\sqrt{n}} - \frac{s^2}{2n} + \frac{s^3}{3n\sqrt{n}} - \frac{s^4}{4n^2} + \dots\right)$$

$$n\log\left(1+\frac{s}{\sqrt{n}}\right) - s\sqrt{n} = n\left(\frac{s}{\sqrt{n}} - \frac{s^2}{2n} + \frac{s^3}{3n\sqrt{n}} - \frac{s^4}{4n^2} + \dots\right) - s\sqrt{n}$$

$$n\log\left(1+\frac{s}{\sqrt{n}}\right) - s\sqrt{n} = \left(s\sqrt{n} - \frac{s^2}{2} + \frac{s^3}{3\sqrt{n}} - \frac{s^4}{4n} + \dots\right) - s\sqrt{n}$$

$$n\log\left(1+\frac{s}{\sqrt{n}}\right) - s\sqrt{n} = -\frac{s^2}{2} + \frac{s^3}{3\sqrt{n}} - \frac{s^4}{4n} + \dots$$

Now taking the limit on both sides as $n \to \infty$,

$$\lim_{n\to\infty} n\log\left(1+\frac{s}{\sqrt{n}}\right) - s\sqrt{n} = -\frac{s^2}{2}$$

From this, we can deduce,

$$n! = n^n\sqrt{n}e^{-n}\int_{-\sqrt{n}}^{\infty} e^{-s\sqrt{n}}e^{n\log\left(1+\frac{s}{\sqrt{n}}\right)}ds$$

$$\frac{n!}{n^n\sqrt{n}e^{-n}} = \int_{-\sqrt{n}}^{\infty} e^{-s\sqrt{n}}e^{n\log\left(1+\frac{s}{\sqrt{n}}\right)}ds$$

$$\lim_{n\to\infty}\frac{n!}{n^n\sqrt{n}e^{-n}} = \int_{-\infty}^{\infty} e^{-s\sqrt{n}}e^{-\frac{s^2}{2}+s\sqrt{n}}ds$$

$$\lim_{n\to\infty}\frac{n!}{n^n\sqrt{n}e^{-n}} = \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}ds$$

$$\lim_{n\to\infty}\frac{n!}{n^n\sqrt{n}e^{-n}} = \sqrt{2\pi}$$

$$\lim_{n\to\infty}\frac{n!}{n^n\sqrt{2\pi n}e^{-n}} = 1$$

□

Lemma 3.1

$$\beta(x,y) \sim \frac{2\sqrt{\pi}x^{x+\frac{1}{2}}y^{y+\frac{1}{2}}}{(x+y)^{x+y+\frac{1}{2}}}$$

Proof:

$$
\begin{aligned}
\beta(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \\
&\approx \frac{x^x\sqrt{2\pi x}e^{-x}y^y\sqrt{w\pi y}e^{-y}}{(x+y)^{x+y}\sqrt{2\pi(x+y)}e^{-x-y}} \\
&\approx \frac{x^{x+\frac{1}{2}}y^{y+\frac{1}{2}}\sqrt{2\pi}}{(x+y)^{x+y+\frac{1}{2}}}
\end{aligned}
$$

Fun fact!

Notice that this approximation is very useful when computen large numbers, since the approximation can be solved in polynomial time.

Corollary 3.2

Use Stirling's formula to show that

$$\lim_{n\to\infty} n^x\beta(x,n) = \Gamma(x)$$

Proof:

$$
\begin{aligned}
n^x\beta(x,n) &= n^x\frac{\Gamma(n)\Gamma(x)}{\Gamma(x+n)} \\
&= n^x\frac{(x+n)(n)}{(x+n)(n)}\frac{\Gamma(n)\Gamma(x)}{\Gamma(x+n)} \\
&= \frac{n^x(x+n)\Gamma(n+1)\Gamma(x)}{n\Gamma(x+n+1)} \\
&\approx \frac{n^x(x+n)n^n e^{-n}\sqrt{2\pi n}\Gamma(x)}{n(x+)^{x+n}e^{-x-n}\sqrt{2\pi(x+n)}} \\
&\approx \frac{e^x\Gamma(x)n^{x+n-\frac{1}{2}}}{(x+n)^{x+n-\frac{1}{2}}}
\end{aligned}
$$

Hence, taking the limit when $n\to\infty$ in both sides,

$$\lim_{n\to\infty} n^x \beta(x,n) = \lim_{n\to\infty} \frac{e^x \Gamma(x) n^{x+n-\frac{1}{2}}}{(x+n)^{x+n-\frac{1}{2}}}$$
$$= e^x \Gamma(x) \lim_{n\to\infty} \left(\frac{n}{x+n}\right)^{x+n-\frac{1}{2}}$$

Now, we have to evaluate the limit on the right.

Notice that,

$$\left(\frac{n}{x+n}\right)^{x+n-\frac{1}{2}} = \left(\frac{n}{x+n}\right)^{n} \left(\frac{n}{x+n}\right)^{x-\frac{1}{2}} = \left(\frac{x+n}{n}\right)^{-n} \left(\frac{x+n}{n}\right)^{\frac{1}{2}-x}$$

$$= \left(1+\frac{x}{n}\right)^{-n} \left(1+\frac{x}{n}\right)^{\frac{1}{2}-x}$$

Evaluating those two limits,

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} = e^{-x}$$

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{\frac{1}{2}-x} = 1$$

Thus, obtaining

$$\lim_{n\to\infty} \left(1+\frac{n}{x+n}\right)^{x+n-\frac{1}{2}} = e^{-x}$$

Therefore,

$$\lim_{n\to\infty} \beta(x,n) = e^x \Gamma(x) \lim_{n\to\infty} \left(\frac{n}{x+n}\right)^{x+n-\frac{1}{2}}$$

$$\lim_{n\to\infty} \beta(x,n) = \Gamma(x)$$

□

6 Further reading

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