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Combinatoria - Profesor Julián Abril
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Capítulo 2 del libro de Chong y Meng, ejercicios 24-31. (28 y 29 ya no).

1. **Ejercicio 24**

$$\sum_{r=0}^{n} 3^r \binom{n}{r} = 4^n$$

Solución:

Usando el teorema binomial, hacemos que $x = 1, y = 3$.

Así,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$(1+3)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 3^r$$

$$4^n = \sum_{r=0}^{n} \binom{n}{r} 3^r$$

2. **Ejercicio 25**

$$\sum_{r=0}^{n} (r+1)\binom{n}{r} = (n+2)2^{n-1}$$

Solución:

Haciendo uso de la identidad $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$, tenemos que:

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$

$$r \cdot \binom{n}{r} = n \cdot \binom{n-1}{r-1}.$$

Reemplazando, obtenemos que:

$$\begin{aligned}
\sum_{r=0}^{n}(r+1)\binom{n}{r} &= \sum_{r=0}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r} \\
&= \sum_{r=0}^{n} n\cdot\binom{n-1}{r-1} + \sum_{r=0}^{n}\binom{n}{r} \\
&= n\sum_{r=0}^{n}\binom{n-1}{r-1} + \sum_{r=0}^{n}\binom{n}{r} \\
&= n2^{n-1} + 2^{n} \\
&= 2^{n-1}(n+2)
\end{aligned}$$

3. **Ejercicio 26**

$$\sum_{r=0}^{n}\frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}(2^{n+1}-1)$$

Solución:

Usando el teorema binomial, hacemos $x = 1$.

Así,

$$(x+y)^{n} = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^{r}$$

$$(1+y)^{n} = \sum_{r=0}^{n}\binom{n}{r}y^{r}$$

Integrando,

$$\frac{1}{n+1}(1+y)^{n+1} + c = \sum_{r=0}^{n}\binom{n}{r}\frac{1}{1+r}y^{r+1}$$

Sea $y = 0$

$$\frac{1}{n+1}(1+0)^{n+1} + c = \sum_{r=0}^{n}\binom{n}{r}\frac{1}{1+r}0^{r+1}$$

$$\frac{1}{n+1} + c = 0$$

$$c = -\frac{1}{n+1}$$

Por lo tanto,

$$\frac{1}{n+1}(1+y)^{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n}\binom{n}{r}\frac{1}{1+r}y^{r+1}$$

Ahora, sea $y = 1$.

$$\frac{1}{n+1}(1+1)^{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r} 1^{r+1}$$

$$\frac{1}{n+1}2^{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r}$$

$$\frac{1}{n+1}(2^{n+1} - 1) = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r}$$

4. **Ejercicio 27**

$$\sum_{r=0}^{n} \frac{(-1)^r}{r+1} \binom{n}{r} = \frac{1}{n+1}$$

Solución:

Usando el teorema binomial, hacemos $x = 1$.

Así,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$(1+y)^n = \sum_{r=0}^{n} \binom{n}{r} y^r$$

Integrando,

$$\frac{1}{n+1}(1+y)^{n+1} + c = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r} y^{r+1}$$

Sea $y = 0$

$$\frac{1}{n+1}(1+0)^{n+1} + c = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r} 0^{r+1}$$

$$\frac{1}{n+1} + c = 0$$

$$c = -\frac{1}{n+1}$$

Por lo tanto,

$$\frac{1}{n+1}(1+y)^{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r} y^{r+1}$$

Ahora, sea $y = -1$.

$$\frac{1}{n+1}((-1)+1)^{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r}(-1)^{r+1}$$

$$\frac{1}{n+1}(0)^{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{1+r}(-1)^{r+1}$$

$$-\frac{1}{n+1} = \sum_{r=0}^{n} \binom{n}{r} \frac{(-1)^{r+1}}{1+r}$$

$$\frac{1}{n+1} = \sum_{r=0}^{n} \binom{n}{r} \frac{(-1)^{r}}{1+r}$$

5. **Ejercicio 28**

$$\sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} = 2^{n-m}\binom{n}{m}$$

Para $m \leq n$.

Solución:

$$\sum_{r=0}^{n} \binom{n}{r}\binom{r}{m} = \sum_{r=0}^{n} \binom{n}{m}\binom{n-m}{r}$$

$$= \binom{n}{m} \sum_{r=0}^{n} \binom{n-m}{r}$$

$$= \binom{n}{m} 2^{n-m}$$

6. **Ejercicio 30**

$$\sum_{r=0}^{m} (-1)^{m-r}\binom{n}{r} = \binom{n-1}{m}$$

Para $m \leq n-1$.

Solución:

Usando el teorema binomial, hacemos $x = -1, y = 1$.

Así,

$$((-1)+1)^n = \sum_{r=0}^{n} \binom{n}{r}(-1)^{n-r}1^r$$

$$0 = \sum_{r=0}^{n} \binom{n}{r}(-1)^{n-r}$$

7. **Ejercicio 31**

$$\sum_{r=0}^{n}(-1)^r r\binom{n}{r} = 0$$

Solución:

Usando el teorema binomial, hacemos $x = 1, y = -1$.

Así,

$$(1+(-1))^n = \sum_{r=0}^{n}\binom{n}{r}1^{n-r}(-1)^r$$

$$0 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r$$

Derivando respecto a r,

$$0 = \sum_{r=0}^{n}\binom{n}{r}r(-1)^{r-1}$$

Multiplicando por -1 en ambos lados,

$$0 = \sum_{r=0}^{n}\binom{n}{r}r(-1)^r$$