Problems for the midterm test

NLA 2017

The midterm test will include both theoretical questions and problems based on the material of the first 9 lectures. Note that during the test you are not allowed to use any source of information, e.g. laptop, handwritten materials or your classmates. Cheating is obviously prohibited.

1 Theoretical questions

Theoretical question can be any question on theory covered in lectures. They do not presume that you give proofs of theorems. The focus will be on your understanding of concepts covered in the course. An example of such questions is "When does the power method fail to converge?".

2 List of problems

The following list of problems will help you prepare for the midterm test. Some part of the problems will be exactly as they are formulated in the list. Others will be with modifications. Also, some problems can be similar to problems from psets 1 and 2.

1. Find $\|F_n\|_2$, $\|F_n\|_F$, $\|F_n\|_1$, where F_n is the Fourier matrix.
2. Suppose you are given a linear model $y = ax + b$ and data points $(x, y) : (0, 1), (1, 2), (2, 4)$. Write down a system on coefficients a and b and find its least squares solution using pseudoinverse.
3. Let $U \in \mathbb{C}^{n\times k}$, $k < n$ matrix so that $U^*U = I_k$. Find a pseudoinverse of UU^*.
4. Find singular values of the matrix

$$A = \begin{bmatrix} 1 \\ 2 \\ \dots \\ n \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}.$$

5. Find gradient and Hessian of $J(x) = \operatorname{trace}(\operatorname{diag}(x) A \operatorname{diag}(x))$.
6. Find the distance between a nonsingular matrix and the closest singular in $\|\cdot\|_2$ norm.
7. Find the distance between a singular matrix and the closest nonsingular in $\|\cdot\|_2$ and $\|\cdot\|_1$ norms.
8. Show that the normalized Fourier matrix is unitary.
9. Find $\operatorname{cond}_\infty \begin{bmatrix} 1 & 2 & & & 0 \\ & 1 & 2 & & \\ & & \ddots & \ddots & \\ & & & 1 & 2 \\ 0 & & & & 1 \end{bmatrix}$.
10. Let $A \in \mathbb{C}^{m\times n}$. Prove that

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|, \quad \|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

11. Decomposition

$$A = H + iK, \quad H = H^*, \quad K = K^*, \quad i^2 = -1$$

is called Hermitian decomposition of A.

   (a) Does it always exist?

   (b) Using Hermitian decomposition prove that if $(Ax, x) \geq 0$ for all $x \in \mathbb{C}^n$, then A is Hermitian.

   (c) Is it true that if $(Ax, x) \geq 0$ for all $x \in \mathbb{R}^n$ then A is Hermitian? Prove or provide a counter example.

12. Show that $\|A\|_F \leq \sqrt{\text{rank}(A)}\|A\|_2$.

13. Prove that $\rho(A) \leq \|A\|_2$ where $\rho(A)$ denotes spectral radius of A.

14. Let $\mu \in \lambda(A + E)$, $\mu \notin \lambda(A)$. Prove that (Bauer-Fike theorem)

$$\frac{1}{\|(A - \mu I)^{-1}\|_2} \leq \|E\|_2.$$

15. Let $A = U\Sigma V^*$ be SVD decomposition of A. Find SVD decomposition of the matrix $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$.

16. Let matrix $A \in \mathbb{C}^{n\times n}$ be given by its skeleton decomposition $A = BC^*$, where $B, C \in \mathbb{C}^{n\times r}$, $r < n$. Suggest an algorithm that finds SVD of A in $\mathcal{O}(nr^2 + r^3)$ operations assuming that B and C are given.

17. Show that strictly diagonally dominant matrices are nonsingular.

18. The goal of compressed sensing is to find the sparsest solution x of an undetermined linear system $y = Ax$ where $A \in \mathbb{R}^{n\times m}, n < m$. In order to achieve it one could try to find solution which has minimal first norm. Intuition behind this fact is quite simple in 2D:

   (a) Draw disks $\|x\| = \text{const}$ for 1, 2 and ∞ norms.

   (b) Find graphically solutions of $y = Ax$, $\|x\|_* \to \min$, where $A \in \mathbb{R}^{1\times 2}$ and $* = \{1, 2, \infty\}$. Which norm yields the sparsest solution?

19. Let $u, v \in \mathbb{R}^{n\times 1}$. Find $\det(I + uv^*)$.