Low-rank Approximations for Incomplete Matrices

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Background

There are some challenging industrial problems in which only incomplete data is available and the goal is to “complete” the data.

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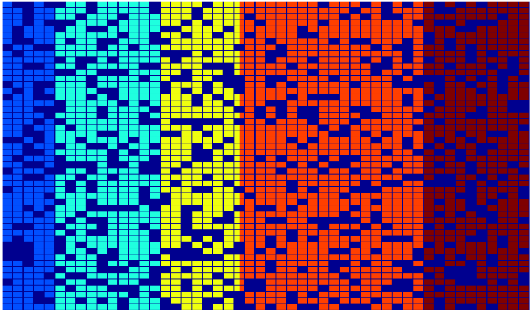

Problem Formulation

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General Formulation

Input:

- Data dimensions (M, N)
- Set of known entries Ω and values at them (sparse matrix X)

Output:

- Low-rank approximation Z

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Approach I

Fix rank, minimize error at the known entries.

Approach II

Set maximum acceptable error at the known entries, minimize rank.

Implemented approaches

- Alternating Least Squares (Approach I)
- Riemannian optimization [Vandereycken, 2012] (Approach I)
- Soft-Input [Mazumder, Hastie, Tibshirani, 2010] (Approach II)

Alternating Least Squares(ALS)

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$$\underset{Z}{\text{minimize}} \quad \frac{1}{2} \sum_{(i,j)\in\Omega} (X_{ij} - Z_{ij})^2 \quad \text{s.t.} \quad \mathrm{rank}(Z) = K$$

Alternating Least Squares (ALS)

- Find Z in the form $Z = U^T V$, $U \in \mathbb{R}^{K\times M}$, $V \in \mathbb{R}^{K\times N}$
- Update U and V independently until convergence
- At each step optimal U and V can be found analytically

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Avoiding overfitting: add regularization term $\lambda(\|U\|_F^2 + \|V\|_F^2)$, still explicit formulas for optimal U and V^T at each step.

Riemannian optimization

- Matrices of fixed-rank k forms a smooth manifold of dimensionality $(m+n-k)k$.
- Tangent space of the same dimensionality.
- Algorithm closely resembles a typical non-linear CG algorithm with Armijo line-search for unconstrained optimization.

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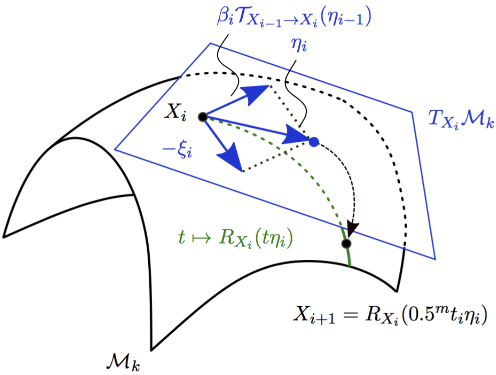
Visualization of non-linear CG on a Riemannian manifold



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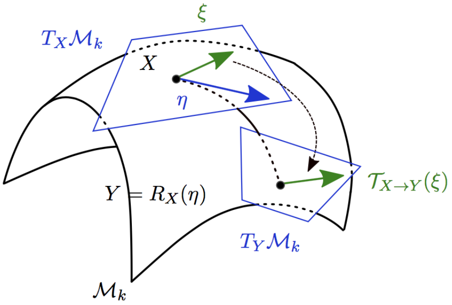
Vector transport on a Riemannian manifold



Initial formulation:

$$\underset{Z}{\text{minimize}} \quad \mathrm{rank}(Z) \quad \text{s.t.} \quad \frac{1}{2} \sum_{(i,j)\in\Omega} (X_{ij} - Z_{ij})^2 \leq \delta \qquad (1)$$

Rank minimization

Initial formulation:

$$\underset{Z}{\text{minimize}} \quad \mathrm{rank}(Z) \quad \text{s.t.} \quad \frac{1}{2} \sum_{(i,j) \in \Omega} (X_{ij} - Z_{ij})^2 \leq \delta \qquad (1)$$

Convex relaxation of (1):

$$\underset{Z}{\text{minimize}} \quad \|Z\|_* \quad \text{s.t.} \quad \frac{1}{2} \sum_{(i,j) \in \Omega} (X_{ij} - Z_{ij})^2 \leq \delta \qquad (2)$$

Rank minimization

Initial formulation:

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Equivalent reformulation of (2):

$$\underset{Z}{\text{minimize}} \quad \frac{1}{2}\sum_{(i,j)\in\Omega}(X_{ij}-Z_{ij})^2 + \lambda\|Z\|_* \qquad (3)$$

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If the full X is known and $U\Lambda V^T$ is SVD for X, then the solution to (3) is given by

$$Z = U\Lambda_\lambda V^T, \text{ where } \Lambda_\lambda = \mathrm{diag}\Big((\sigma_1 - \lambda)_+, \ldots, (\sigma_{\min\{M,N\}} - \lambda)_+\Big).$$

SOFT-INPUT

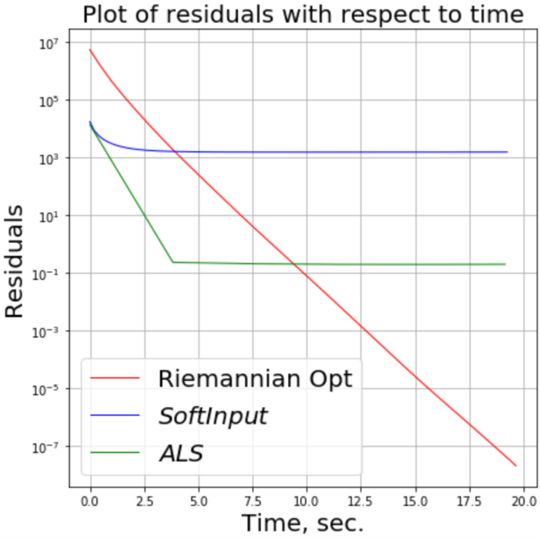
- Set Z_0 to be a zero-matrix.
- At each step k approximate unknown entries of X by Z_{k-1}, set Z_k to be a solution for a problem with all values given.

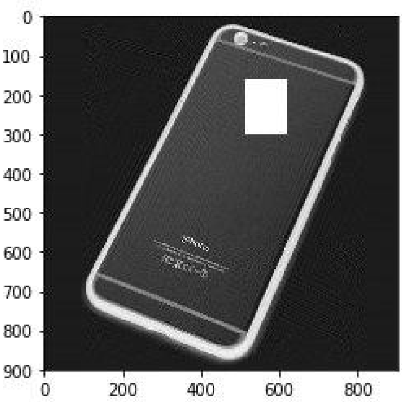
NOTE: Since approximated full matrix is of form $Z_k + (X - Z_{\Omega,k})$, MATVEC can be cheap.

Comparative Analysis

	ALS	**Riemannian**	**SOFT-INPUT**
Parallelizable	✓	✓	✗
Adaptive rank choice	✗	✗	✓
Flops per iteration	$O(\lvert\Omega\rvert K^2 + K^3)$	$O((M+N)K^2 + \lvert\Omega\rvert K))$	$O((M+N+\lvert\Omega\rvert)K^3)$

Comparative Analysis

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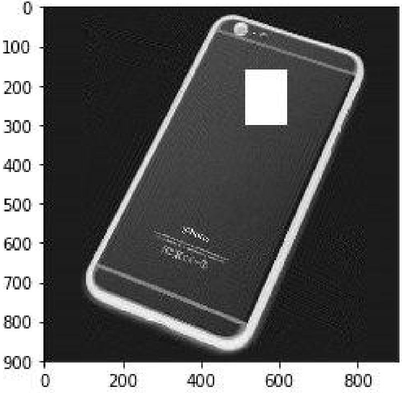
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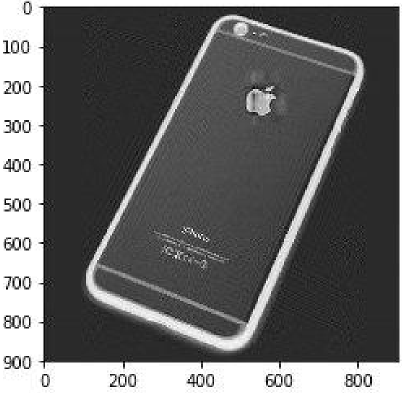
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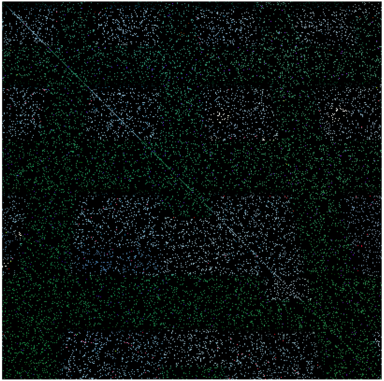
Demonstration



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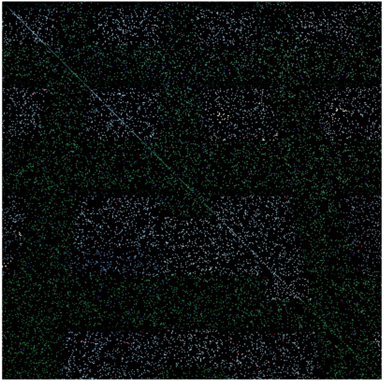

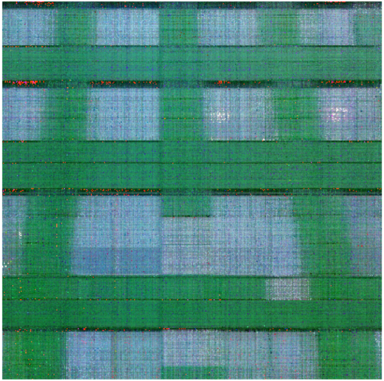

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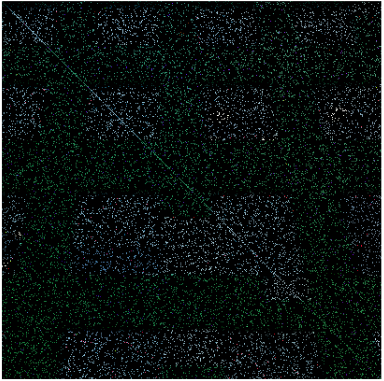
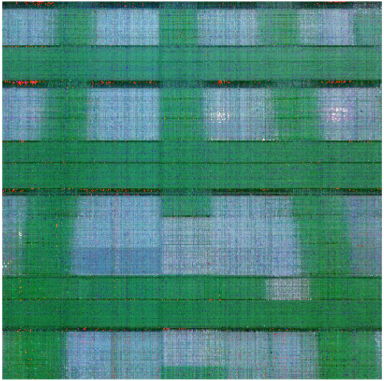


75 kB



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- Implementation of three competitive completion algorithms
- Comparative analysis of implemented algorithms
- Application to repairing damaged files