Lecture 2. Supplementary materials

1. Chebyshev norm is not submultiplicative

$$\|A\|_c \overset{def}{=} \max_{i,j} |a_{ij}| \text{ — Chebyshev norm}$$

$$\|AB\|_c \not\leq \|A\|_c \cdot \|B\|_c$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \Rightarrow \|AB\|_c = 2$$

$$\|A\|_c = \|B\|_c = 1$$

$$\|AB\|_c = 2 > 1 = \|A\|_c \cdot \|B\|_c$$

2. Frobenius norm is not an operator norm

$$\underset{\sqrt{n}}{\overset{\|}{\|I\|_F}} = \sup_{x \neq 0} \frac{\|Ix\|_*}{\|x\|_*} = 1$$

For $\sup \frac{\| \ \|_*}{\| \ \|_{**}}$ not obvious

3. $\|A\|_\infty$

$$\|A\|_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sup_{x \neq 0} \frac{\max_i |\sum_j a_{ij} x_j|}{\max_i |x_i|} \leq$$

$$\leq \sup_{x \neq 0} \frac{\cancel{\max_i |x_i|} \cdot \max_i \sum_j |a_{ij}|}{\cancel{\max_i |x_i|}} = \max_i \sum_j |a_{ij}| = C$$

$$\|A\|_\infty \leq C$$

$x_0 = (\mathrm{sign}(a_{i_0 1}), \mathrm{sign}(a_{i_0 2}), \dots, \mathrm{sign}(a_{i_0 n}))^T$:

$\frac{\|Ax_0\|_\infty}{\|x_0\|_\infty} = C$

i_0 — index, on which the maximum is achieved

4. Unitary invariance of vector norm $\|\cdot\|_2$

$$\|Uz\|_2^2 = \|z\|_2^2 \iff U \text{ is unitary } UU^* = U^*U = I$$

$\Leftarrow$: $\|Uz\|_2^2 = \langle Uz, Uz\rangle = (Uz)^* Uz = z^* \underbrace{U^*U}_{I} z = z^*z = \|z\|_2^2$

$\Rightarrow$: homework

5. Unitary invariance of $\|\cdot\|_2$.

$$\|\underbrace{UAV}_{\text{unitary}}\|_2 = \sup_{x\neq 0} \frac{\|UAVx\|_2}{\|x\|_2} \overset{\substack{\|\cdot\|_2\\ \text{unit. inv}}}{=} \sup_{x\neq 0} \frac{\|A\overbrace{Vx}^{y}\|_2}{\|x\|_2} =$$

$$= \sup_{V^{-1}y\neq 0} \frac{\|Ay\|_2}{\|V^{-1}y\|_2} = \sup_{y\neq 0} \frac{\|Ay\|_2}{\|V^*y\|} = \|A\|_2$$

$\|V^*y\| = \|y\|_2$

$$VV^* = I \iff V^* = V^{-1}$$

6. Permutation matrix

$$P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$P_3 P_3^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

7. Housholder matrix

$$H^*H = (I - 2vv^*)^*(I - 2vv^*) = (I - 2vv^*)^2$$

$$\left[(vv^*)^* = v^{**}v^* = vv^*\right]$$

$$= I - 2vv^* - 2vv^* + 4v\underbrace{v^*v}_{1}v^* = I$$

$$\exists v:\ H(v)\overset{\overset{x}{\shortparallel}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$