Review of Holographic Second Laws for Conformal Field Theories Out of Equilibrium

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Overview

- Review of the α-relative Rényi entropies and the 2^{nd} laws of thermodynamics
- Conformal field theories (CFTs) & correspondence to anti-de Sitter (AdS)
  - Renormalisation group (RG), c-theorem, and CFTs
- Applying α-RRE to CFTs
  - Path integral for a quenched state
  - Implications for RG flows

Relative Rényi Entropies

- Quantum relative divergence (QRD): $S(\rho||\sigma) \coloneqq \mathrm{Tr}\left[\rho(\ln\rho - \ln\sigma)\right]$
  - Standard measure of distinguishability between two states. Generalises von Neumann.
- By analogy with Rényi entropy, generalisable to one-parameter family?
  - Proper[1] and standard: for $\alpha \in [0, 1]$, define[2] quantum Rényi entropy (QRE) as:

$$S_\alpha(\rho||\sigma) \coloneqq \frac{1}{1-\alpha} \ln \frac{\mathrm{Tr}\left[\rho^\alpha \, \sigma^{(1-\alpha)}\right]}{\mathrm{Tr}\,\rho}$$

  - Extension[3,4] to $\alpha \in [1, \infty)$: sandwiched relative Rényi entropy (sandwiched-RRE).

$$S_\alpha(\rho||\sigma) \coloneqq \frac{1}{1-\alpha} \ln \frac{\mathrm{Tr}\left[\left(\sigma^{(1-\alpha)/2\alpha} \, \rho \, \sigma^{(1-\alpha)/2\alpha}\right)^\alpha\right]}{\mathrm{Tr}\,\rho}$$

[1] – F. Hiai and D. Petz, Commun. Math. Phys. **143**, 99 (1991).
[2] – H. Umegaki, Kodai Math. Sem. Rep. **14**, 59 (1962).
[3] – M. Müller-Lennert *et al.*, arXiv:1306.3142.
[4] – M. M. Wilde, A. Winter, and D. Yang, Comm. Math. Phys. **331**, 593 (2014).

α-Free Energies

- Textbook free energy: $F(\rho) = \langle E \rangle_\rho - TS(\rho) = \langle H \rangle_\rho - T\,\mathrm{Tr}\,[\rho \ln \rho]$
  - Transitions from ρ_i to ρ_f only for $F(\rho_f) \leq F(\rho_i)$; i.e. if $\Delta F \coloneqq F(\rho_f) - F(\rho_i) \leq 0.$
- Brandão et al.[5]: generalise to family of α-free energies:

$$F_\alpha(\rho) \coloneqq -k_B T \ln Z + k_B T\, S_\alpha(\rho || \rho_G)$$

  - We can do this due to monotonicity of Rényi divergences: generalises 2nd Law.
  - Transitions from ρ_i to ρ_f only for $F_\alpha(\rho_f) \leq F_\alpha(\rho_i)$ for <u>all</u> α.
  - $S_\alpha(\rho || \rho_G)$: relative entropy between ρ and ρ_G (thermal state).
- Question: what new insights can this give for conformal field theories?

[5] – F. G. S. L. Brandão *et al.*, PNAS **112**, 3275 (2015).

CFT and AdS Correspondence

- Quantum field theories invariant under conformal transformations:

$$g_{\mu\nu}(x^\sigma) \mapsto f(x^\sigma)\, g_{\mu\nu}(x^\sigma)$$

  - $f(x^\sigma) \in \mathbb{R}^+$ is the scale factor, and must be positive definite.
- Scale-invariance: CFTs are fixed points of RG flows.
  - QFTs are either CFTs (at a fixed point) or are at specific points in RG flows between fixed points.
- Maldacena[6]: gauge theories in an AdS space have a duality to a given CFT.
  - AdS: spacetime with constant negative curvature.
  - $SO(2, n-1)$ symmetry: isomorphic to conformal group in $(n-1)$ dim.

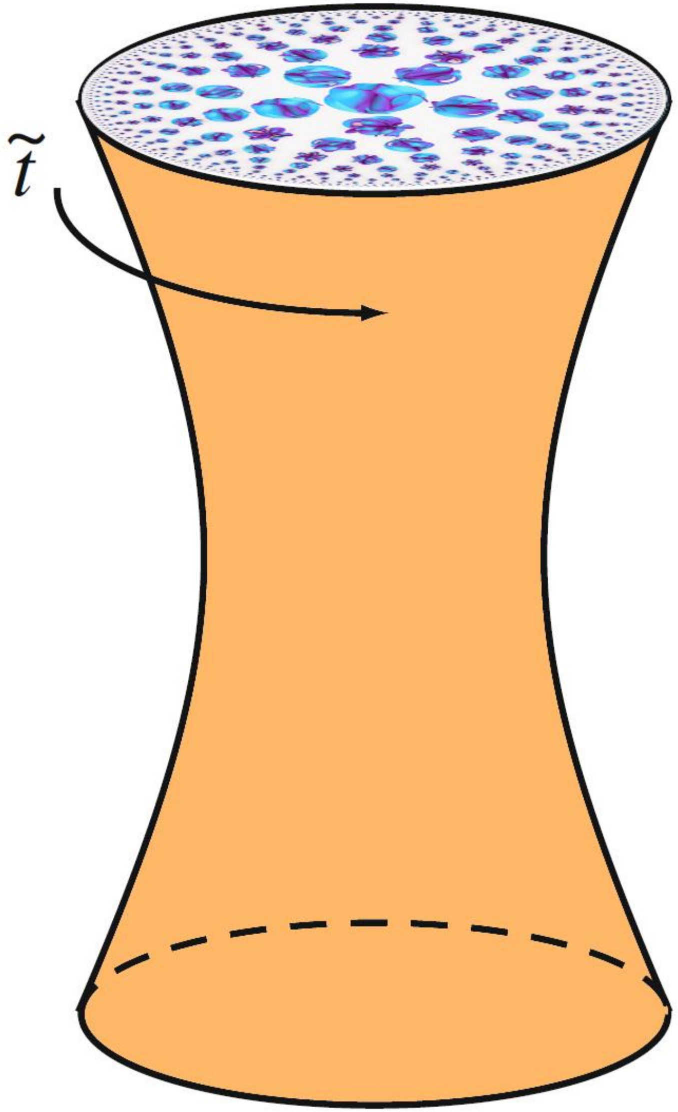


Image modified from M. Natsuume, *AdS/CFT Duality User Guide* (Springer Nature, Tokyo, 2015), via Glen Faught.

[6] – J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

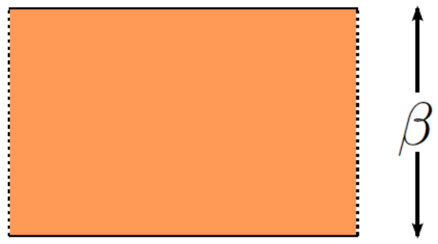
Path Integral for Quenched Excited States

- In terms of Cauchy surfaces, a thermal state is[7] straightforward:

$$\rho_\beta = e^{-\beta H_{\mathrm{CFT}}} =$$

- Similarly, a globally excited state at a specific point in time is simply:

$$\rho_\beta = e^{-\beta H_{\mathrm{CFT}}} = \int [D\phi]\, e^{-S_{\mathrm{CFT}}[\phi] - \eta \int \mathrm{d}^d x\, \mathcal{O}_\Delta(x)} =$$



  - $\mathcal{O}_\Delta(x)$ is an operator of conformal dimension Δ that generates the excited state.
    - Conformal dimension: scale exponent $\mathcal{O} \mapsto \lambda^{-\Delta}\mathcal{O}$ under dilations $x \mapsto \lambda x$.
  - η is the amplitude of the excitation.

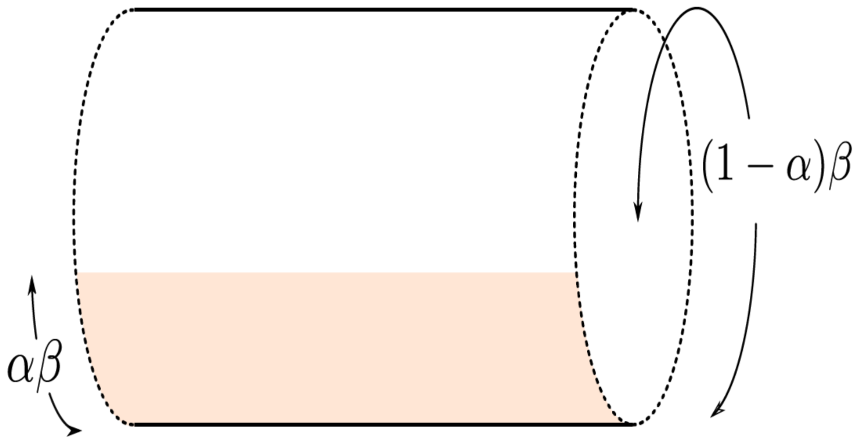
Images from A. Bernamonti, F. Galli, R. C. Meyers, and J. Oppenheim, J. High Energy Phys. **2018**, 111 (2018).

[7] – A. Bernamonti, F. Galli, R. C. Meyers, and J. Oppenheim, J. High Energy Phys. **2018**, 111 (2018).

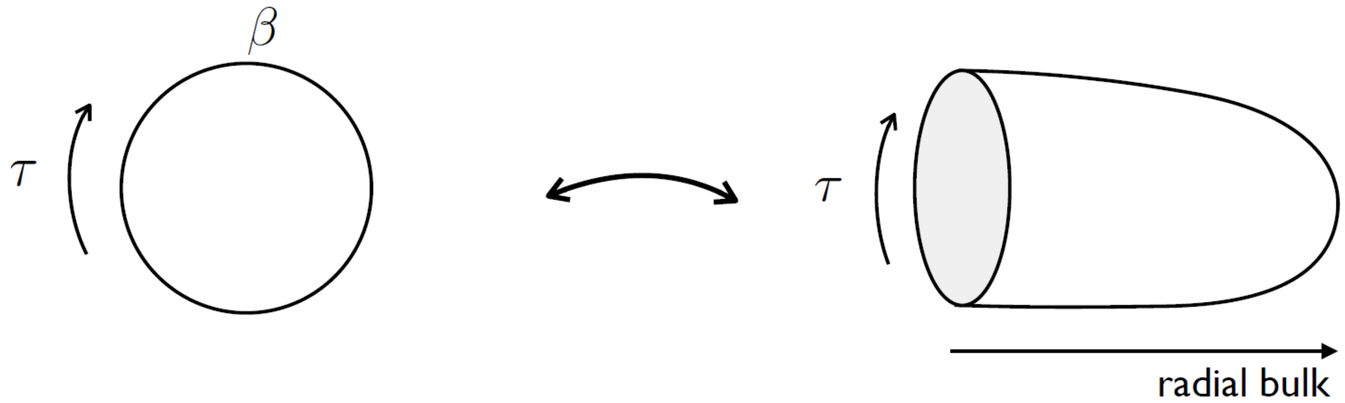
Quenched State Relative Entropies

- To calculate $\mathbf{Tr}\left[\rho^{\alpha}\,\rho_G^{\;(1-\alpha)}\right]$, BGMO[7] stitch these together:

$$\mathbf{Tr}\left[\rho^{\alpha}\,\rho_G^{\;(1-\alpha)}\right] =$$



- AdS/CFT dictionary relates a finite temperature CFT to a black hole state in an AdS space:



First image from A. Bernamonti *et al.*, J. High Energy Phys. **2018**, 111 (2018).
Second image from F. Galli (unpublished) [talk at PASCOS 2018 associated with this paper].

[7] – A. Bernamonti, F. Galli, R. C. Meyers, and J. Oppenheim, J. High Energy Phys. **2018**, 111 (2018).

Renormalisation Group

- RG tells influence of coupling constants (CCs) in Hamiltonian over length scales.
  - $H(\{K_a\})$. Examine $K_a \to K_a + \delta\ell\,(\partial K_a/\partial\ell)$. Defines <u>beta function</u>: $\beta(K_a) \coloneqq \partial K_a/\partial\ell$.
  - Fixed point: couplings invariant under renormalisations. ($\beta^a = \partial\{K_a\}/\partial\ell = 0.$)
  - Ising model: $H = -K\sum_{\langle i,j\rangle}\sigma_i\sigma_j$. Examine how K changes as we coarse-grain.
    - Ising: $H \to -K_1\sum_{\langle i,j\rangle}\sigma_i\sigma_j - K_2\sum_{\langle\langle i,j\rangle\rangle}\sigma_i\sigma_j - K_2\sum_{\langle\langle i,j\rangle\rangle}\sigma_i\sigma_j\sigma_k\sigma_l + \cdots$

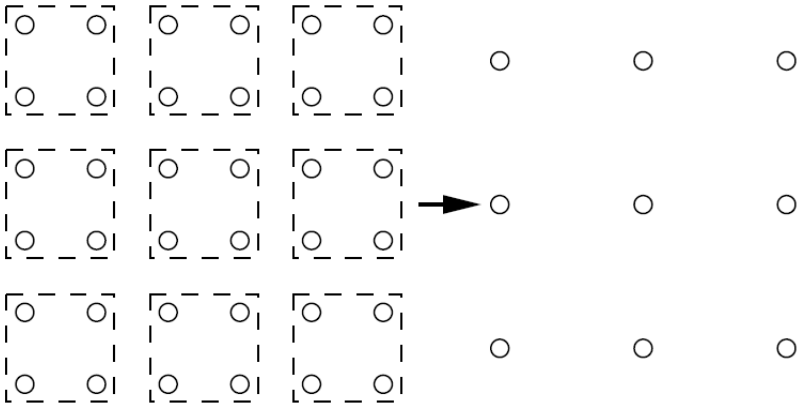
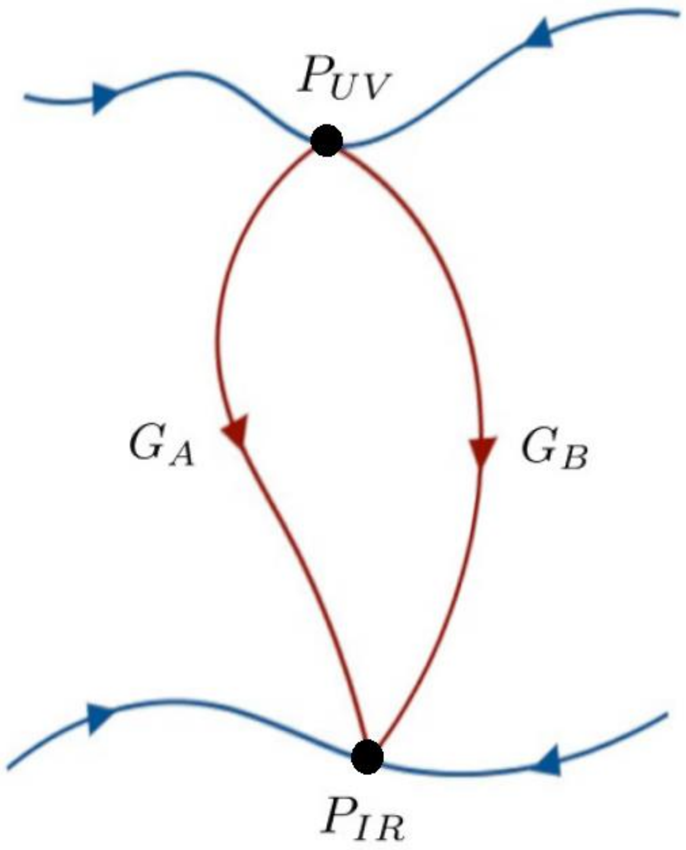


Image from W. D. McComb, *Renormalization Methods: A Guide for Beginners* (Oxford University Press, Oxford, 2004).

c-Theorem

- In CC space, $\boldsymbol{H}$ represented by points $\vec{K} \in \mathbb{R}^a$.
  - RG maps points to other points.
  - RG flow given by eigenvalues of RG transformation.
- Zamolodchikov[8]: fixed points are CFTs!
  - β-function: velocity in CC space. $\mathbf{d/d}t_K \coloneqq -\beta^a(K)\ \partial/\partial\{K_a\}$
  - c-function: monotone decreasing along RG flow. $\mathbf{d}C/\mathbf{d}t_K \leq \mathbf{0}.$
  - At fixed point, c-function yields central charge of a CFT.
- QFTs are points on RG flow between CFTs.
- Gives direction to RG flow: $c_{\mathrm{IR}} \leq c_{\mathrm{UV}}$.



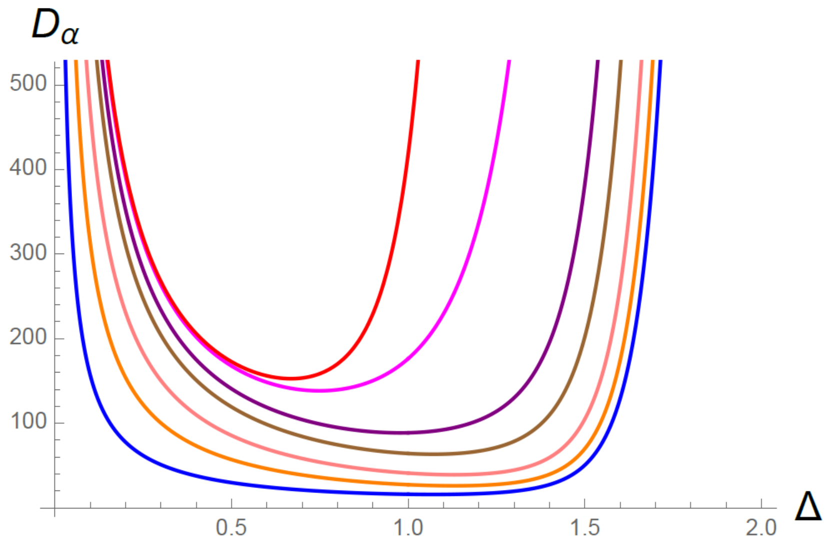
[8] – A. B. Zamolodchikov, JETP Lett. **43**, 730 (1986).
Image from H. Casini, R. Medina, I. S. Landea, and G. Torroba, J. High Energy Phys. **2018**, 166 (2018).

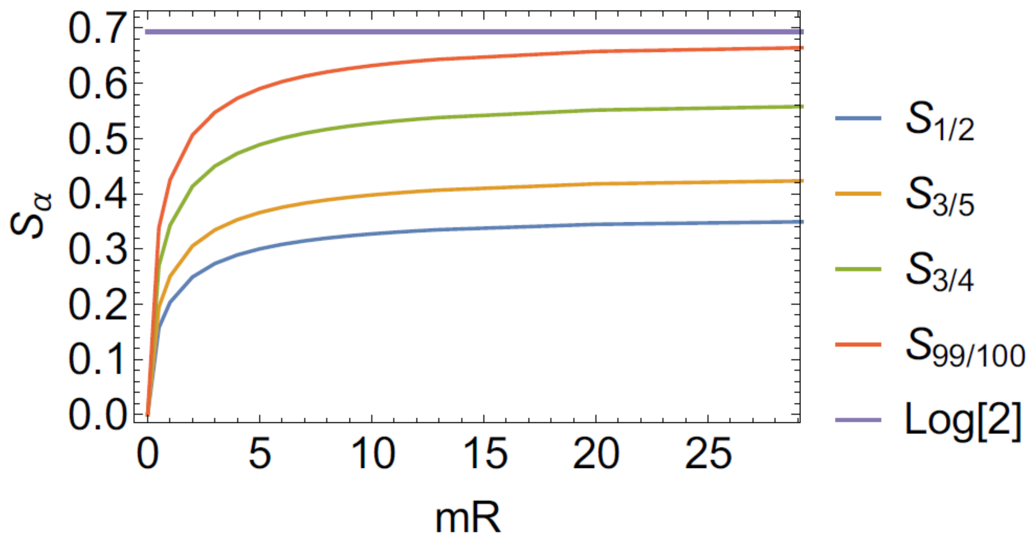
Results for CFTs

- BGMO[7]: α-free energies yield extra restrictions on transitions!
- CMLT[9]: extra restriction on RG trajectories:

$$S_\alpha(\rho_{\Delta,\mathrm{UV}} \| \rho_{G,\mathrm{UV}}) \leq (c_{\mathrm{UV}} - c_{\mathrm{IR}}) \ln \Lambda_0 R/3$$

- Ugajin[10]: preliminary relation of S_α to quasiparticle evolution[11] in CFTs.





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Left image from [7]. Right image from [9].

Conclusions and Next Steps

- α-second laws offer powerful new constraints on conformal field theories.
  - In quenched excited state, <u>explicit</u> case of transitions forbidden by $\alpha \neq 1$ second laws.
  - Stronger bounds on transitions between CFTs, including $\alpha \neq 1$ forbidden transitions.
  - Can be used to examine set of all possible allowed evolutions of quasiparticles.
- Next steps
  - Application to the entanglement wedge reconstruction.
  - Application to <u>specific</u> CFTs, as well as fuller examination of applications to CFT quasiparticles.
    - Can be used to provide bounds relevant for examining topological quantum computation out of equilibrium.

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- You, for your attention! Thank you!

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α-RREs, α-z-RREs, and Thermodynamic Second Laws

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Abstract

The discovery of a family of second laws of thermodynamics based on the α-RRE provides a powerful tool to examine transitions out of equilibrium, in particular yielding a full family of constraints on the types of possible transitions available to a given system. Meanwhile, the discovery of the AdS/CFT correspondence gave rise to the holographic entanglement entropy, a unique and rich way of examining the (equilibrium) entanglement entropies of states in boundary CFTs in terms of the shared boundary surface areas.

A natural question to investigate nonequilibrium properties of boundary CFTs, then, is to investigate the effect that nonequilibrium techniques of calculating entropies can have on the properties of the holographic entanglement entropy. In this talk, I'll review properties of the holographic entanglement entropy that might be unfamiliar to some, and then discuss recent applications of the second laws of thermodynamics to the holographic entanglement entropy. In particular, I'll discuss the origin of the holographic entanglement entropy from the AdS/CFT correspondence, recent techniques used to examine α-RREs of excited states of CFTs, and recent results on the properties of mixed states in CFTs.

von Neumann, Rényi, and Relative Entropies

- von Neumann (vN) entropy of state with density matrix ρ: $S(\rho) = \mathrm{Tr}\,[\rho \ln \rho]$
- Generalise vN to 1-parameter family of entropies: Rényi entropies.

$$S_\alpha(\rho) := \frac{1}{1-\alpha} \ln \frac{\mathrm{Tr}\,[\rho^\alpha]}{\mathrm{Tr}\,\rho}$$

  - Serves as a unifying family for various entropies: vN, min., max., etc.
- Generalise vN to a measure of distinguishability between two states: quantum relative divergence (QRD).

$$S(\rho||\sigma) := \mathrm{Tr}\,[\rho(\ln \rho - \ln \sigma)]$$

Relative Rényi Entropy Candidates

- Is QRD generalisable to one-parameter family?
  - Quantum Rényi entropy as previously defined is the standard.
- Problem: ∞ possible ways[12] of arranging $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ to get a valid expression.
  - Example 1: sandwiched RRE[3,4]:

$$S_\alpha(\rho\|\sigma) \coloneqq \frac{1}{1-\alpha}\ln\frac{\mathrm{Tr}\left[\left(\sigma^{(1-\alpha)/2\alpha}\,\rho\,\sigma^{(1-\alpha)/2\alpha}\right)^{\alpha}\right]}{\mathrm{Tr}\,\rho}$$

  - Example 2: reversed-sandwiched RRE[12]:

$$S_\alpha(\rho\|\sigma) \coloneqq \frac{1}{\alpha-1}\ln\frac{\mathrm{Tr}\left[\left(\rho^{\alpha/2(1-\alpha)}\,\sigma\,\rho^{\alpha/2(1-\alpha)}\right)^{(1-\alpha)}\right]}{\mathrm{Tr}\,\rho}$$

[3] – M. Müller-Lennert *et al.*, arXiv:1306.3142.
[4] – M. M. Wilde, A. Winter, and D. Yang, Comm. Math. Phys. **331**, 593 (2014).
[12] – K. M. R. Audenaert and N. Datta, J. Math. Phys. **56**, 022202 (2015).

α-z-RRE

- Problem: ρ and σ don't commute, so there's an infinite number of ways of slicing up powers of them to get a valid Rényi entropy.
- In order to encompass all possible ways we can do this, define[12] a two-parameter family: the α-z-relative Rényi entropy. (α-z-RRE)

$$S_{\alpha,z}(\rho||\sigma) := \frac{1}{\alpha - 1} \ln \frac{\mathrm{Tr}\left[\left(\rho^{\alpha/z}\,\sigma^{(1-\alpha)/z}\right)^{z}\right]}{\mathrm{Tr}\,\rho}$$

- Specific Rényi entropies of interest are then specific cases of this.

[12] – K. M. R. Audenaert and N. Datta, J. Math. Phys. **56**, 022202 (2015).

α-z-RRE as an Encompassing Framework

- α-z-RRE encompasses all possible quantum entropies:
  - Reduces to QRE for z = 1.
  - Reduces to sandwiched RRE for z = α.
  - Reduces to reversed-sandwiched RRE for z = 1 − α.
  - Yet more possibilities! Examples:

$$S_{\alpha,2} = \frac{1}{\alpha - 1} \ln \frac{\mathrm{Tr}\left[\rho^{\alpha/2}\, \sigma^{(1-\alpha)/2}\, \rho^{\alpha/2}\, \sigma^{(1-\alpha)/2}\right]}{\mathrm{Tr}\, \rho}$$

$$S_{\alpha,\infty} = \lim_{z\to\infty} \frac{1}{\alpha - 1} \ln \frac{\mathrm{Tr}\left[\exp\{\alpha \ln \rho + (1-\alpha) \ln \hat{\sigma}\}\right]}{\mathrm{Tr}\, \rho}$$

Limiting Cases to Retrieve What We Want

- Familiar entropies serve as the limiting cases of various relative entropies derived from α-z-RRE:
  - QRD: $S_1(\rho\|\sigma) \coloneqq \mathrm{Tr}\,[\rho(\ln\rho - \ln\sigma)]$ is $\alpha \to 1$ limit of QRE and sandwiched RRE:

$$\mathrm{Tr}\,[\rho(\ln\rho - \ln\sigma)] = \lim_{\alpha \nearrow 1} \frac{1}{1-\alpha} \ln \frac{\mathrm{Tr}\left[\rho^{\alpha}\,\sigma^{(1-\alpha)}\right]}{\mathrm{Tr}\,\rho} = \lim_{\alpha \searrow 1} \frac{1}{1-\alpha} \ln \frac{\mathrm{Tr}\left[\left(\sigma^{(1-\alpha)/2\alpha}\,\rho\,\sigma^{(1-\alpha)/2\alpha}\right)^{\alpha}\right]}{\mathrm{Tr}\,\rho}$$

  - 0-RRE: $S_0(\rho\|\sigma) \coloneqq -\ln \underset{\{\mathrm{supp}\,\rho\}}{\mathrm{Tr}} \sigma$ is $\alpha \to 0$ limit of reversed-sandwiched RRE:

$$-\ln \underset{\{\mathrm{supp}\,\rho\}}{\mathrm{Tr}} \sigma = \lim_{\alpha \nearrow 1} \frac{1}{1-\alpha} \ln \frac{\mathrm{Tr}\left[\rho^{\alpha}\,\sigma^{(1-\alpha)}\right]}{\mathrm{Tr}\,\rho}$$

More Limiting Cases

- More limits:
  - Min-RRE: $S_{\min}(\rho\|\sigma) = -2\ln\left\|\sqrt{\rho}\sqrt{\sigma}\right\|$ is $\alpha = 1/2$ value of sandwiched-RRE:

$$-2\ln\left\|\sqrt{\rho}\sqrt{\sigma}\right\| = \frac{1}{1-\alpha}\ln\frac{\operatorname{Tr}\left[\left(\sigma^{(1-\alpha)/2\alpha}\,\rho\,\sigma^{(1-\alpha)/2\alpha}\right)^{\alpha}\right]}{\operatorname{Tr}\rho}\Bigg|_{\alpha=1/2}$$

  - Max-RRE: $S_{\max}(\rho\|\sigma) \coloneqq \inf\left\{\lambda\in\mathbb{R}\middle|\rho\le e^{\lambda}\sigma\right\}$ is $\alpha\to\infty$ limit of sandwiched RRE:

$$\inf\left\{\lambda\in\mathbb{R}\middle|\rho\le e^{\lambda}\sigma\right\} = \lim_{\alpha\to\infty}\frac{1}{1-\alpha}\ln\frac{\operatorname{Tr}\left[\left(\sigma^{(1-\alpha)/2\alpha}\,\rho\,\sigma^{(1-\alpha)/2\alpha}\right)^{\alpha}\right]}{\operatorname{Tr}\rho}$$

- Conclusion: α-z-RRE will give us every possible relative entropy, but we need to figure out which z is appropriate.

α-z-RRE Zoo

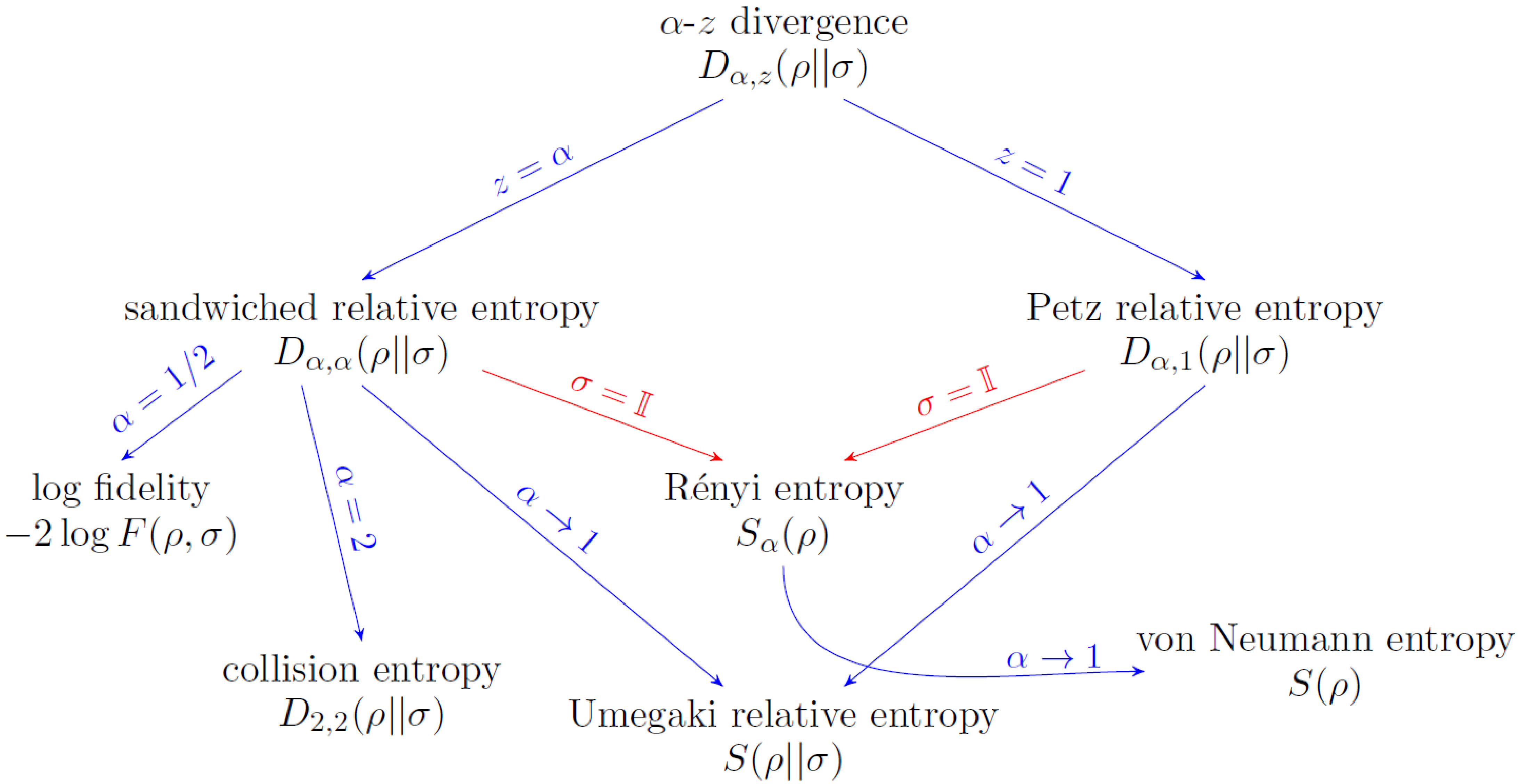


Image from A. May and E. Hijano, J. High Energy Phys. 2018, 36 (2018), simplified from P. Faist (unpublished)

Conformal Field Theories

- A QFT that is invariant under conformal transformations:

$$g_{\mu\nu}(x^\sigma) \mapsto f(x^\sigma)\, g_{\mu\nu}(x^\sigma)$$

  - $f(x^\sigma) \in \mathbb{R}^+$ is the scale factor, and must be positive definite.
  - Poincaré group is subgroup of conformal group, since $f = 1$.
  - Invariance: conformal trans. preserve angles & unitarity.

- Scale-invariance: CFTs are fixed points of RG flows.
  - Systems near critical point; effective theories on topological systems; theories without mass parameters in $\mathcal{L}$.
  - QFTs are either CFTs (at a fixed point) or are at specific points in RG flows between fixed points.

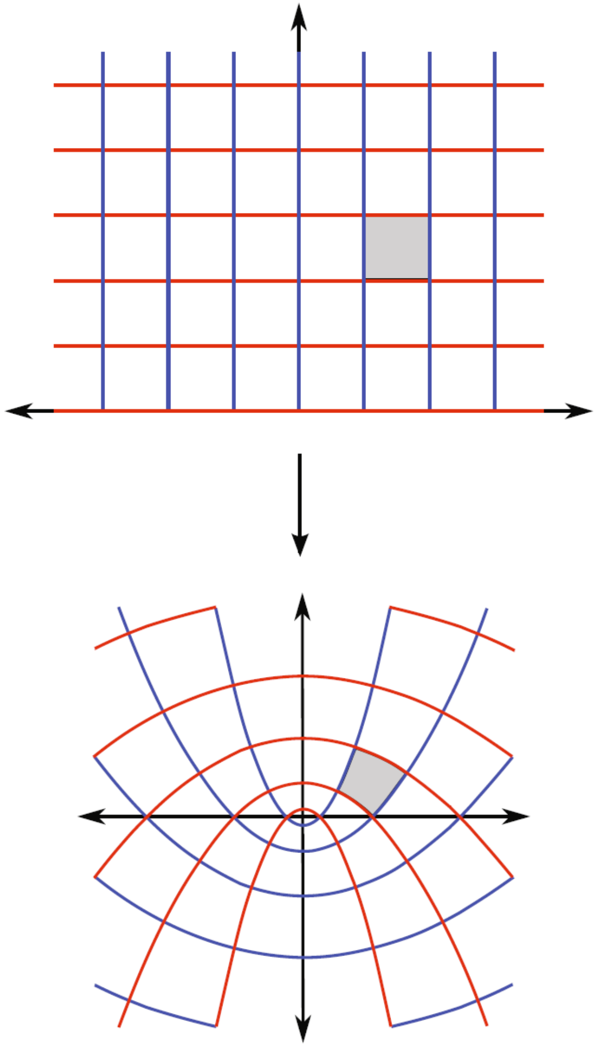


Image from R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory with Applications to String Theory* (Springer Nature, Berlin, 2009).

Anti-de Sitter Space

- Generated by Einstein-Hilbert action with constant negative cosmological constant:

$$\mathcal{L}_{\mathrm{AdS}_n} = \left(g^{\mu\nu} R_{\mu\nu} - \Lambda\right)/16\pi G_{(n)}$$

  - $R_{\mu\nu}$ is Ricci curvature; $G_{(n)}$ is n-dimensional gravitational constant.
- Solution: hyperboloid in $(n-1)$ dimensions.
  - Constant negative curvature.
  - Photons redshifted as they travel away from centre.
  - Trajectories always periodic.
  - Distance to edge is infinite, but photons reach in finite time.
- Exhibits $\mathrm{SO}(2, n-1)$ symmetry.

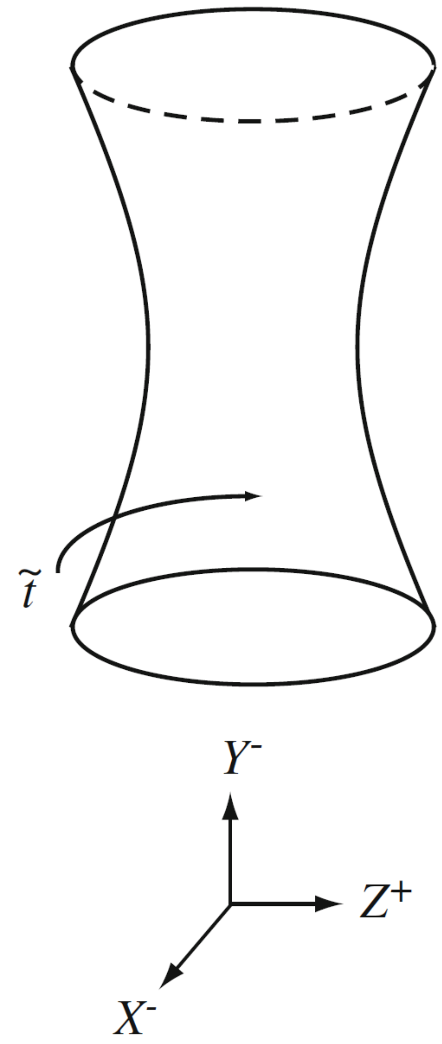
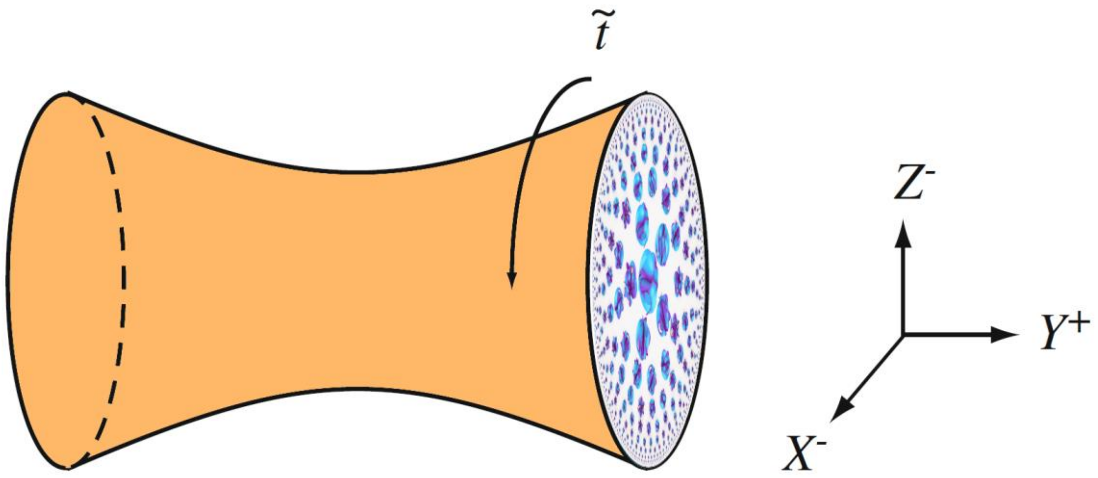


Image from M. Natsuume, *AdS/CFT Duality User Guide* (Springer Nature, Tokyo, 2015)

AdS/CFT Correspondence

- Anti-de Sitter space: hyperboloid in n dimensions.
  - Metric: $\mathrm{d}s^2 = -L^2(\tilde{r}^2+1)\,\mathrm{d}\tilde{t}^2 + \mathrm{d}r^2/(\tilde{r}^2+1) + L^2\,\tilde{r}^2\,\mathrm{d}\Omega_{n-2}^2$
  - Has $\mathrm{SO}(2, n-1)$ symmetry: isomorphic to conformal group in $(n-1)$ dimensions!
- GKP-Witten relation: $Z_{\mathrm{AdS}} = Z_{\mathrm{CFT}}$.
  - Central conjecture of AdS/CFT.
  - Need a dictionary to map quantum gravity in AdS$_d$ to CFT on $\mathbb{R}^{n-1}$



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AdS/CFT Dictionary

Bulk: n-dimensional AdS space (AdS$_n$)	**Boundary: CFT on ∂AdS$_n$ ($=\mathbb{R}^{n-1}$)**
Isometry group $\mathrm{SO}(2, n-1)$	Conformal group $\mathrm{Conf}(\mathbb{R}^{1,n}) \cong \mathrm{SO}(2, n-1)$
Asymptotically locally AdS space	Interacting QFT, with an RG fixed point. • CFT at fixed point must be same as BCFT on ∂AdS$_n$.
States with black holes	Finite-temperature states
Internal gauge symmetry of quantum gravity	Global symmetry
(Scalar) field ϕ: $$\phi(\tilde{r}, \tilde{t}, \Omega) = \int \mathrm{d}\vartheta\, \mathrm{d}\tau\, K\big(\tilde{\theta}, \tilde{r}; \vartheta, \tau\big)\, \mathcal{O}(\tau, \vartheta)$$	Conformal operator $\mathcal{O}$ with dimension Δ: $$\mathcal{O}(\tilde{t}, \Omega) = \lim_{r\to\infty} r^{\Delta}\, \phi(\tilde{r}, \tilde{t}, \Omega)$$
Linear combinations of states and operators, and their respective spaces: $\lvert\psi\rangle_{\mathrm{AdS}} \in \mathcal{H}_{\mathrm{AdS}}$; fields ϕ_{AdS}, $\big(A_\mu\big)_{\mathrm{AdS}}$, $\big(h_{\mu\nu}\big)_{\mathrm{AdS}}$	Linear combinations of states and operators, and their respective spaces: $\lvert\psi\rangle_{\mathrm{CFT}} \in \mathcal{H}_{\mathrm{CFT}}$; fields $\mathcal{O}_{\mathrm{CFT}}$, $\big(J_\mu\big)_{\mathrm{CFT}}$, $\big(T_{\mu\nu}\big)_{\mathrm{CFT}}$

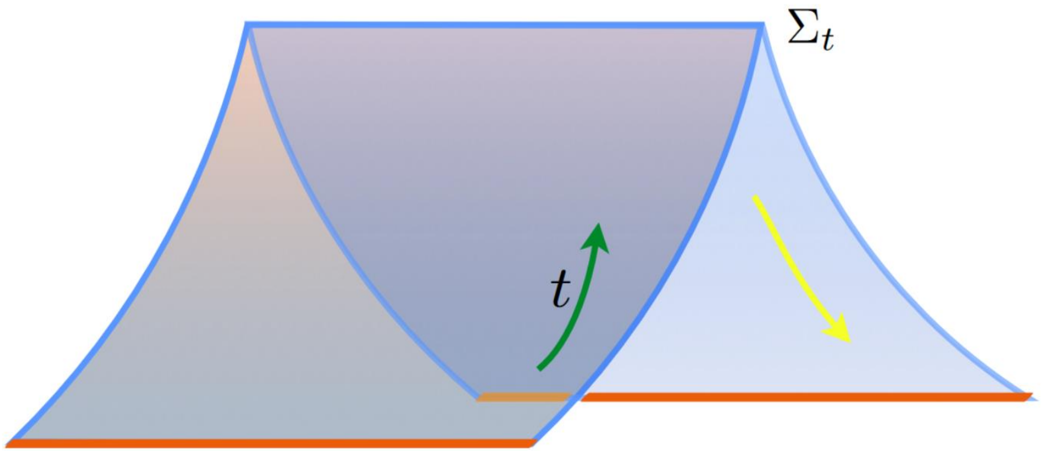
General Features of Path Integrals for S_α

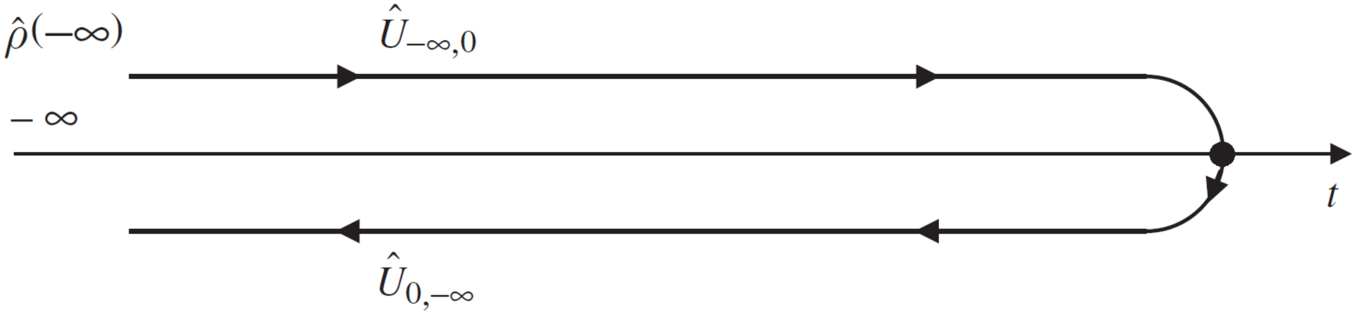
- ρ and $f(\rho, \sigma)$ calculated[13] via Euclidean path integral over a Cauchy surface, e.g.:

$$\rho = \int [\mathcal{D}\phi]\, e^{-S[\phi]}$$

$$\left(\sigma^{(1-\alpha)/_{2\alpha}}\, \rho\, \sigma^{(1-\alpha)/_{2\alpha}}\right)_{ij} = \int [\mathcal{D}\phi][\mathcal{D}a][\mathcal{D}b]\, \langle j|\sigma^{(1-\alpha)/2\alpha}|a\rangle\langle a|\rho|b\rangle\langle b|\sigma^{(1-\alpha)/2\alpha}|i\rangle$$

- Time evolution of fields over Schwinger-Keldysh (in-in) contour.





Left image from X. Dong, A. Lewkowycz, and M. Rangamani, J. High Energy Phys. **2016**, 28 (2016).
Right image modified from A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, Cambridge, 2011).

[13] – X. Dong, A. Lewkowycz, and M. Rangamani, J. High Energy Phys. **2016**, 28 (2016).

Replica Trick

- Can simplify calculating expressions like ρ^{α} by considering these as repeated copies of the Cauchy surface integral for ρ.
  - Sheets are connected by branch cuts, due to imposition of boundary conditions on fields.

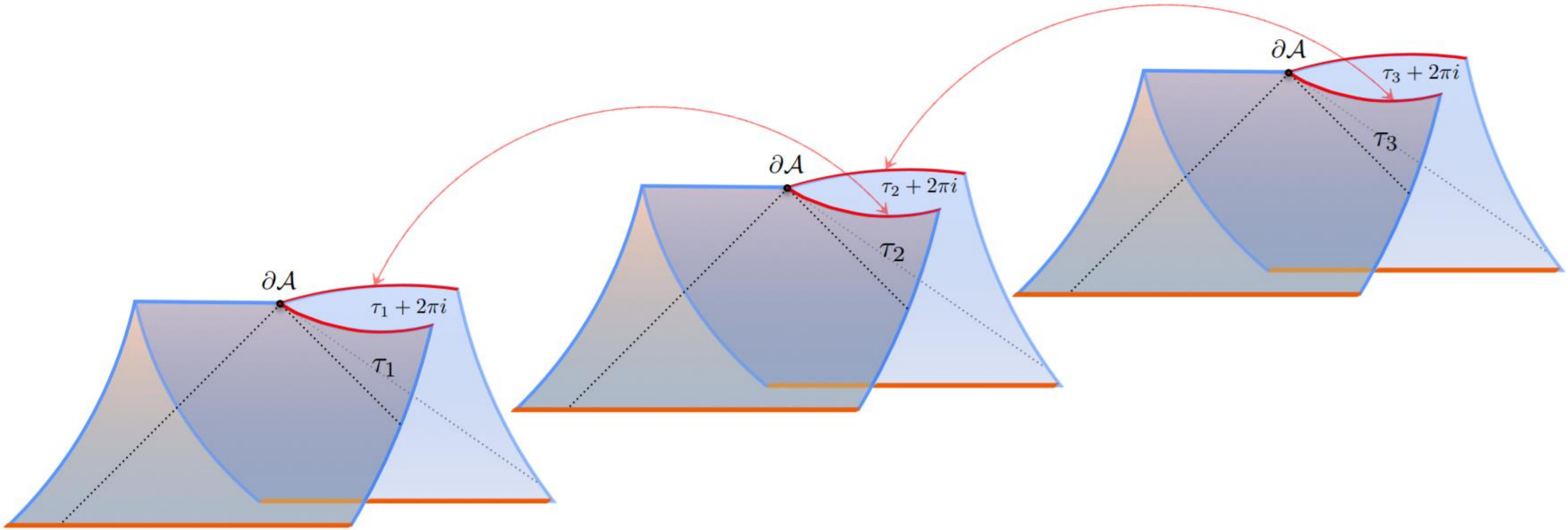


Image from X. Dong, A. Lewkowycz, and M. Rangamani, J. High Energy Phys. **2016**, 28 (2016).

Entanglement Entropy in QFT

- Examine QFT on spacelike slice (Cauchy surface); divide region into $\mathcal{A}$ and $\mathcal{A}^c$.
  - $\partial\mathcal{A}$ is entangling surface.
  - Integrate out $\mathcal{A}^c$ degrees of freedom by defining domain of support (BCs) for $\mathcal{A}$.
    - Insert δ-functional into $\int[\mathcal{D}\phi] : \rho = \int[\mathcal{D}\phi]\, e^{-S[\phi]}\, \delta(\phi_{\mathcal{A}}(t_{0_-}) - \phi_-)\, \delta(\phi_{\mathcal{A}}(t_{0_+}) - \phi_+)$

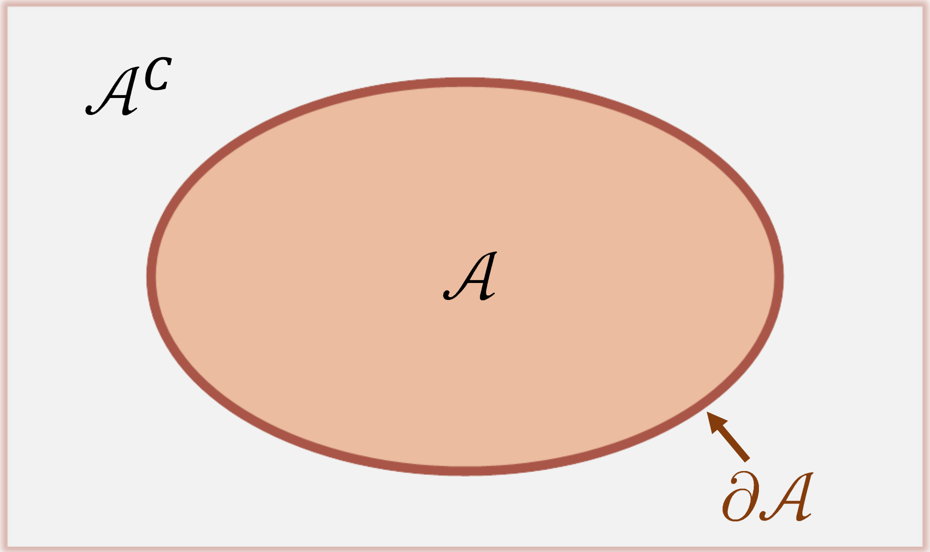


Image modified from M. Rangamani (unpublished) [talk at Quantum Matter, Spacetime, and Information associated with X. Dong, A. Lewkowycz, and M. Rangamani, J. High Energy Phys. **2016**, 28 (2016)].

UV Divergence of QFT Entanglement Entropies

- Short-range correlations cut by entangling surface: UV divergence.
  - Dependent on area of $\partial\mathcal{A}$. UV divergence of entropy $S_{\mathcal{A}}$ has area-law entanglement:

$$S_{\mathcal{A}} = c_{d-2}\left(\frac{L}{\varepsilon}\right)^{d-2} + c_{d-4}\left(\frac{L}{\varepsilon}\right)^{d-4} + \cdots = \gamma\,\frac{\mathbf{A}[\partial\mathcal{A}]}{\varepsilon^{d-2}} + \cdots$$

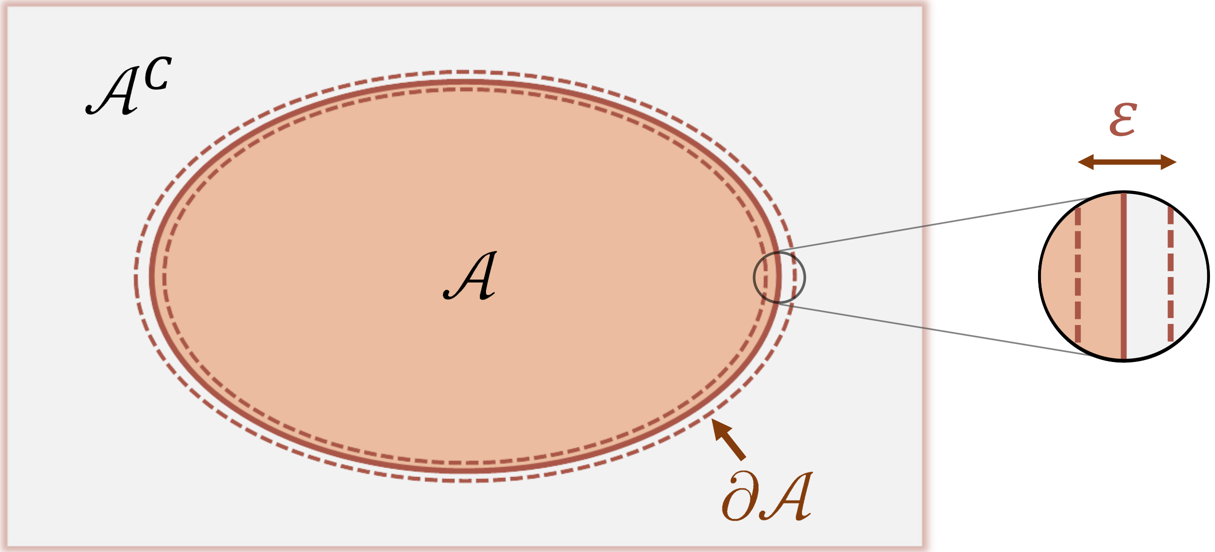


Image modified from M. Rangamani (unpublished) [talk at Quantum Matter, Spacetime, and Information associated with X. Dong, A. Lewkowycz, and M. Rangamani, J. High Energy Phys. **2016**, 28 (2016)].

Subleading Terms

- Universal properties of underlying QFT captured in subleading term $\text{સ}_{\partial\mathcal{A}}$:

$$S_{\mathcal{A}} = \begin{cases} c_{d-2}(L/\varepsilon)^{d-2} + c_{d-4}(L/\varepsilon)^{d-4} + \cdots + c_1(L/\varepsilon) + (-1)^{(d-1)/2}\,\text{સ}_{\partial\mathcal{A}} + \mathcal{O}(\varepsilon) \\ c_{d-2}(L/\varepsilon)^{d-2} + c_{d-4}(L/\varepsilon)^{d-4} + (-1)^{(d-2)/2}\,\text{સ}_{\partial\mathcal{A}}\ln(L/\varepsilon) + \mathcal{O}(\varepsilon) \end{cases}$$

  - Most other coefficients c_i are RG scheme dependent.

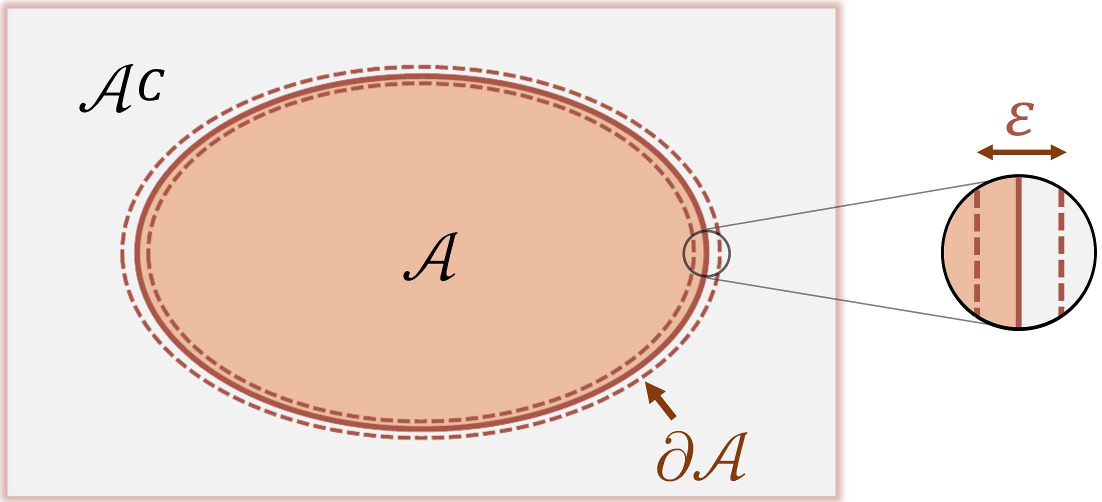


Image modified from M. Rangamani (unpublished) [talk at Quantum Matter, Spacetime, and Information associated with X. Dong, A. Lewkowycz, and M. Rangamani, J. High Energy Phys. **2016**, 28 (2016)].

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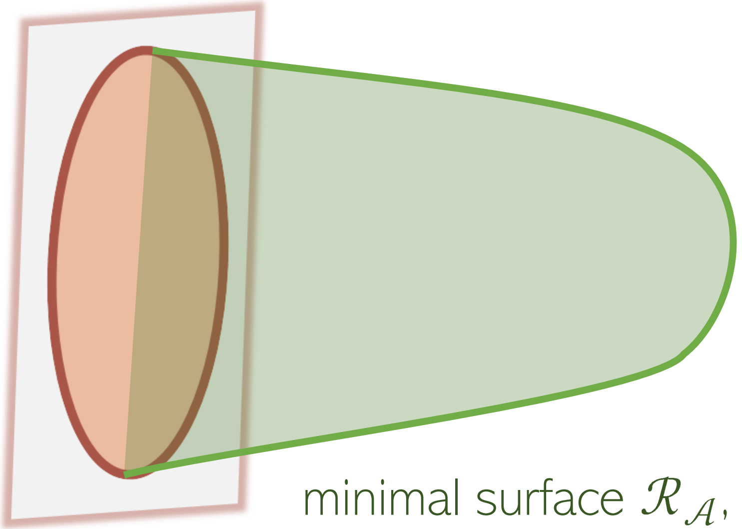
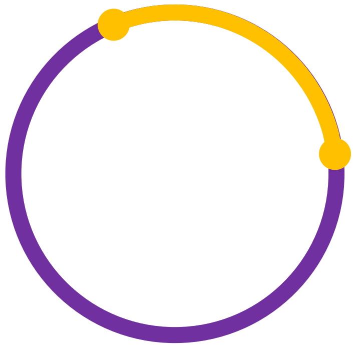
- RT[14] and HRT[15]: leading term found by minimising over surfaces $\mathcal{R}_{\mathcal{A}}$ homologous to $\mathcal{A}$:

$$S_{\mathcal{A}} = \min_{\partial\mathcal{R}_{\mathcal{A}}} \frac{\mathbf{A}(\partial\mathcal{R}_{\mathcal{A}})}{4G_{(n)}} + \cdots \simeq \frac{L^{d-1}}{G_{(n)}} \frac{\mathbf{A}(\partial\mathcal{A})}{\varepsilon^{d-2}} + \cdots$$



  - Surfaces live in the (AdS) bulk.
  - Reproduces area law.
  - Reproduces Calabrese-Cardy result[16] for AdS_3/CFT_2 with charge c

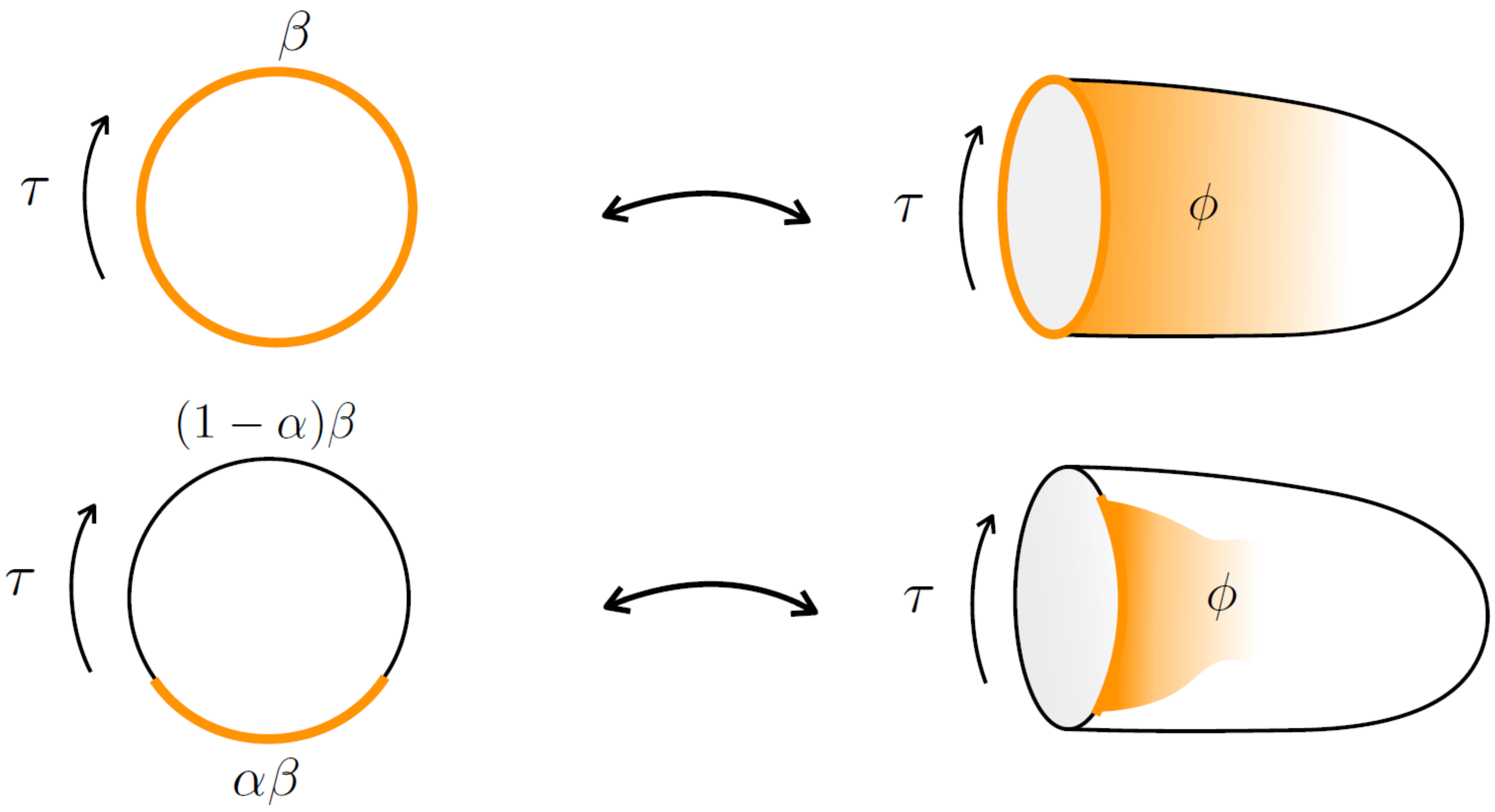
$$S = \frac{c}{3} \ln\left(\frac{\mathcal{C}}{\pi\varepsilon} \sin\frac{\pi\ell}{\mathcal{C}}\right)$$



[14] – S. Ryu and T. Takayanagi, J. High Energy Phys. **2006**, 45 (2006).
[15] – V. E. Hubeny, M. Rangamani, and T. Takayanagi, J. High Energy Phys. **2007**, 62 (2007).
[16] – P. Calabrese and J. Cardy, J. Stat. Mech. **406**, 2 (2004).

Holography for Quenched State CFT

- Similarly, the dictionary relates CFT with an excited state to a scalar field in AdS, and a quenched state to a scalar field with time-dependent boundary conditions:



Renormalisation Group Flow

- In CC space, $\boldsymbol{H}$ represented by points $\vec{K} \in \mathbb{R}^a$. RG maps points to other points.
  - RG flow given by eigenval. of linearised matrix transformation on $\boldsymbol{H}$ close to fixed point.
    - Eigenval. > 1: away from the fixed point. Relevant operator.
    - Eigenval. < 1: towards the fixed point. Irrelevant operator.
    - Eigenval. = 1: need higher-order terms or an alternate approach to analyse. Marginal operator.

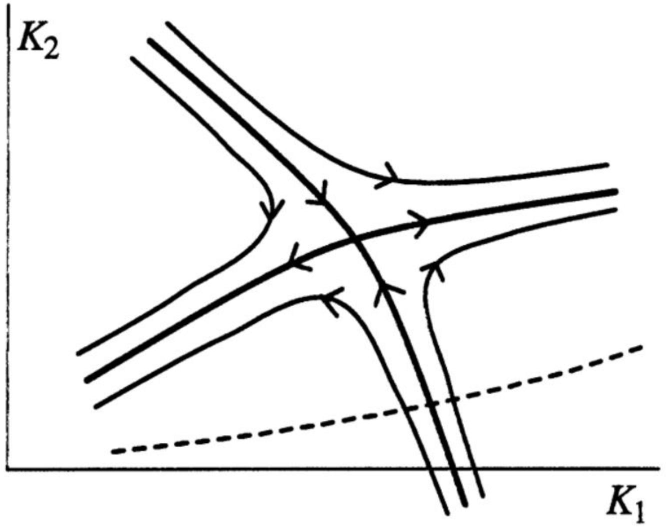


Image from J. Cardy, *Scaling and Renormalization in Statistical Physics* (Oxford University Press, Oxford, 2004).