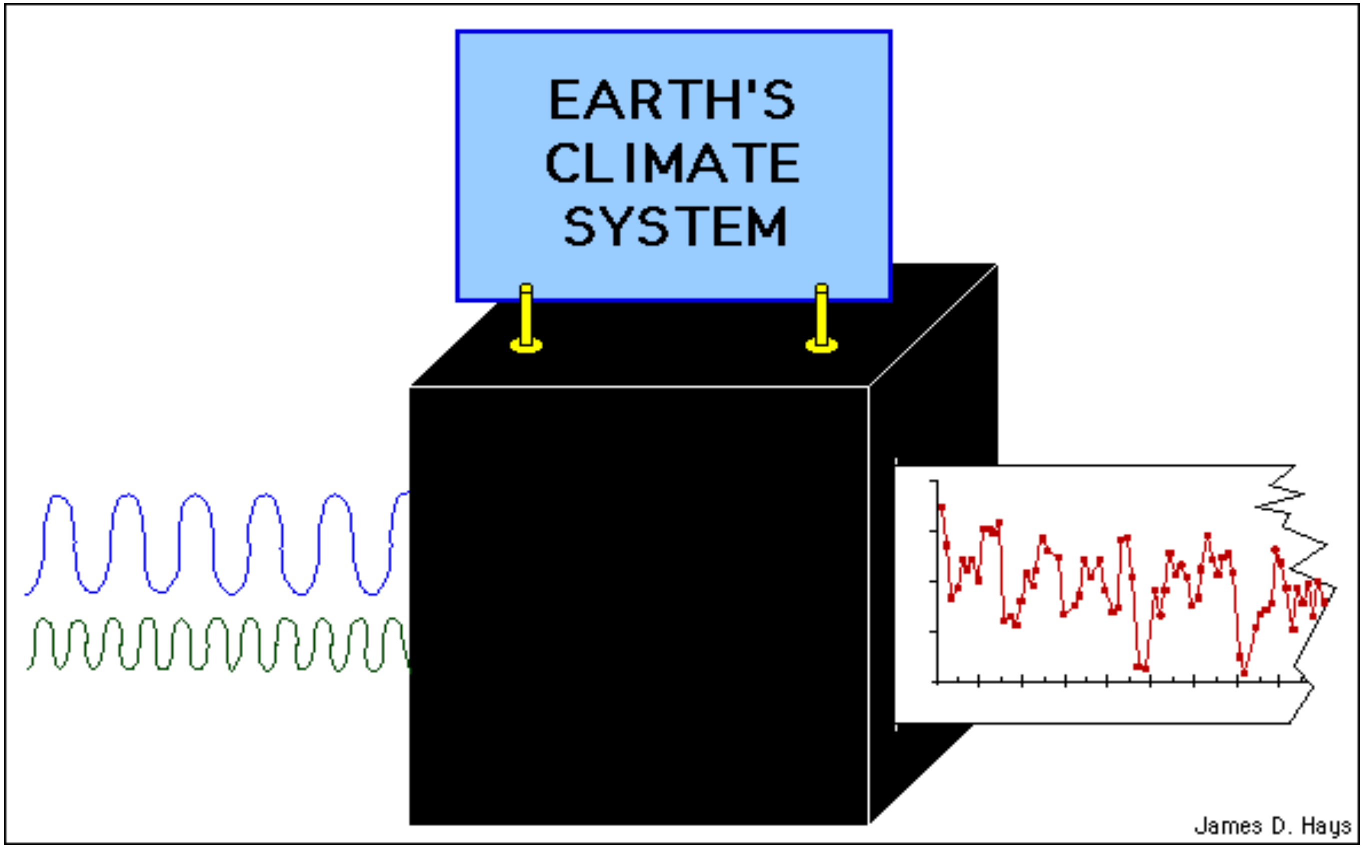
Basics of Dynamical Systems Theory



Navid & Henk, ANU, Jan 20-29, 2020

Numerical Bifurcation Theory

System of PDEs:

$$\mathcal{M}\frac{\partial \mathbf{u}}{\partial t} + \mathcal{L}\mathbf{u} + \mathcal{N}(\mathbf{u}) = \mathcal{F}$$

Operators containing parameters

Discretization (N)

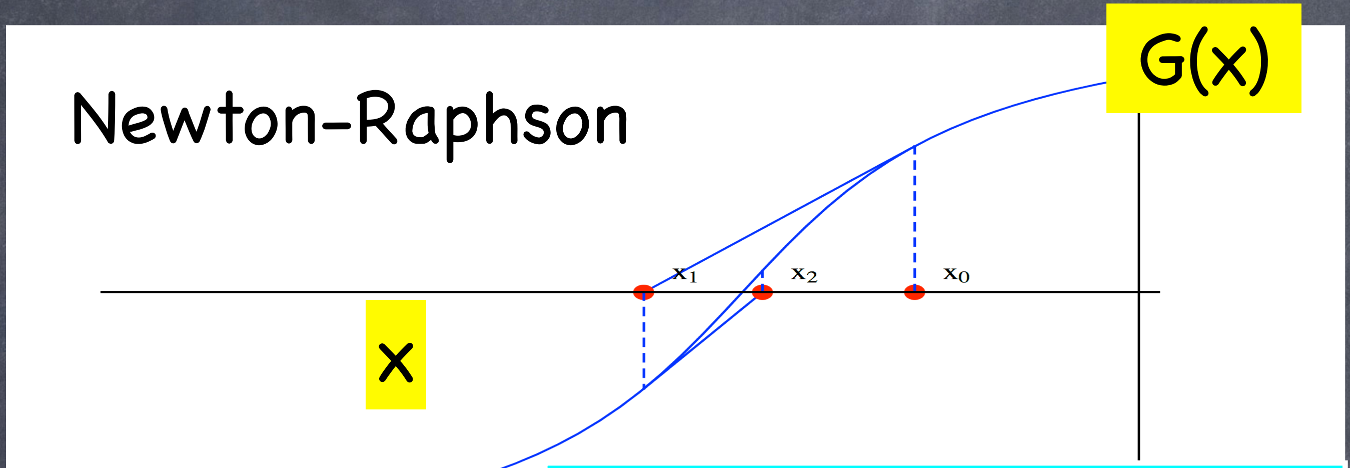
Dynamical system:

$$\mathcal{M}_N\frac{d\mathbf{x}}{dt} + \mathcal{L}_N\mathbf{x} + \mathcal{N}_N(\mathbf{x}) = \mathcal{F}_N$$

x: state vector

The Newton - Raphson process

Scalar function: $G(x) = 0$



$$G(x) = 0 \Rightarrow G'(x^k)\Delta x^{k+1} = -G(x^k)$$

$y = G'(x^k)x + b$ and hence $G(x^k) = G'(x^k)x^k + b$

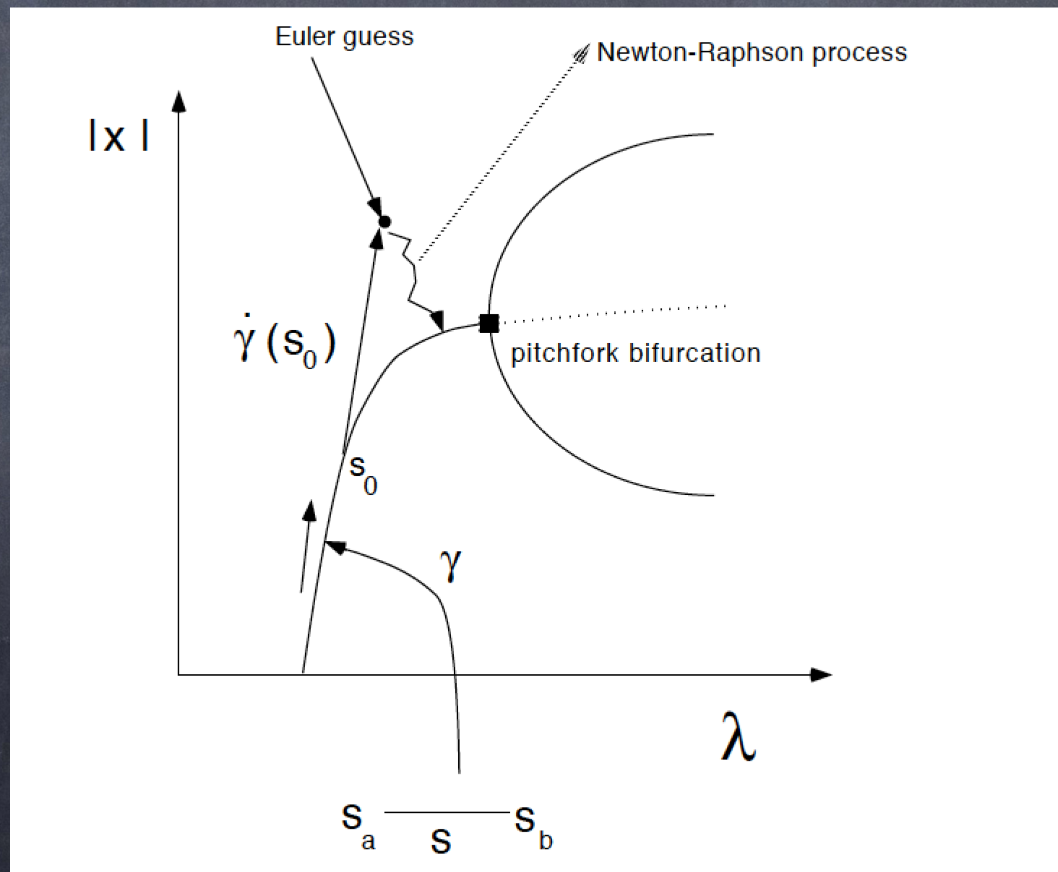
$$\Rightarrow \quad y = G'(x^k)(x - x^k) + G(x^k)$$

Then: $0 = G'(x^k)(x^{k+1} - x^k) + G(x^k)$

Autonomous systems: fixed points

$$\Phi(\mathbf{x}, \lambda) = \mathcal{L}_N \mathbf{x} + \mathcal{N}_N(\mathbf{x}) - \mathcal{F}_N = 0$$

Arclength parametrization



Euler-Newton continuation

Starting Point: $(\mathbf{x}_0, \lambda_0)$

Compute initial tangent: $\dot{\gamma}(s_0) = (\dot{\mathbf{x}}(s_0),\ \dot{\lambda}(s_0))^T$

Solve Extended system: $\Phi(\gamma(s)) = 0$

$$\Sigma(\mathbf{x}, \lambda, s) = \dot{\mathbf{x}}_0^T(\mathbf{x} - \mathbf{x}_0) + \dot{\lambda}_0(\lambda - \lambda_0) - (s - s_0)$$

With Euler guess:

$$\mathbf{x}^1 = \mathbf{x}_0 + \Delta s \ \ \dot{\mathbf{x}}_0$$
$$\lambda^1 = \lambda_0 + \Delta s \ \ \dot{\lambda}_0$$

The initial tangent

Differentiate: $\Phi(\gamma(s)) = 0$ to s:

$$[\Phi_x \;\; \Phi_\lambda]\dot\gamma(s) = \begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1} & \cdots & \frac{\partial \Phi_1}{\partial x_N} & \frac{\partial \Phi_1}{\partial \lambda} \\ \frac{\partial \Phi_1}{\partial x_N} & \cdots & \frac{\partial \Phi_N}{\partial x_N} & \frac{\partial \Phi_N}{\partial \lambda} \end{pmatrix} \dot\gamma(s) = 0$$

If $(\mathbf{x}_0, \lambda_0)$ is not a bifurcation point, then this matrix has rank N

First, the matrix $[\Phi_x \; \Phi_\lambda]$ is triangulated into the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

$\mathbf{v} = (\dot{\mathbf{x}}_0, \dot\lambda_0)$ can be computed by solving

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The Newton - Raphson process for pseudo-arclength continuation

Solve Extended system: $\Phi(\gamma(s)) = 0$

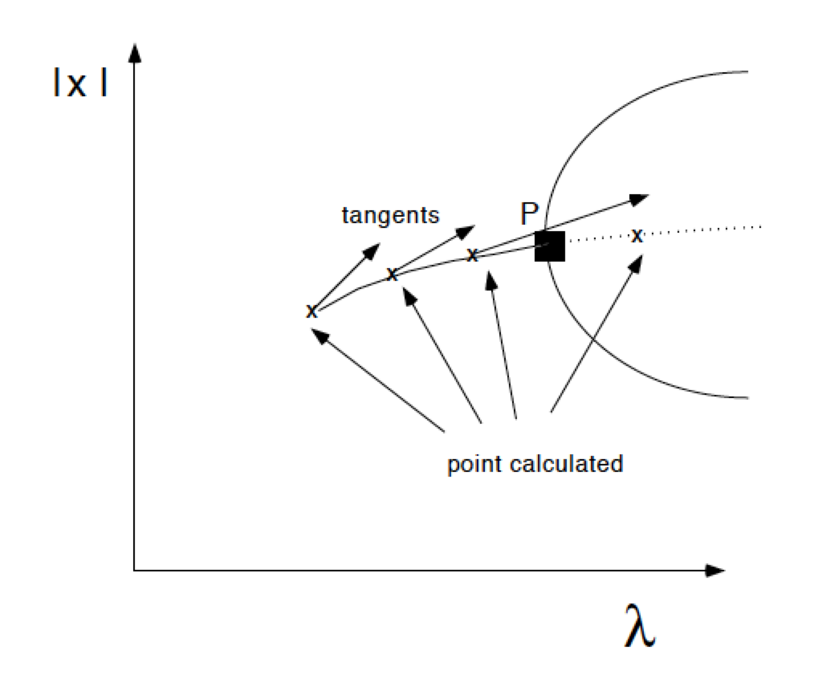
$$\Sigma(\mathbf{x}, \lambda, s) = \dot{\mathbf{x}}_0^T(\mathbf{x} - \mathbf{x}_0) + \dot{\lambda}_0(\lambda - \lambda_0) - (s - s_0)$$

NR-process:

$$\begin{pmatrix} \Phi_x(\mathbf{x}^k, \lambda^k) & \Phi_\lambda(\mathbf{x}^k, \lambda^k) \\ \dot{\mathbf{x}}_0^T & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \Delta \mathbf{x}^{k+1} \\ \Delta \lambda^{k+1} \end{pmatrix} =$$

$$= \begin{pmatrix} -\Phi(\mathbf{x}^k, \lambda^k) \\ \Delta s - \dot{\mathbf{x}}_0^T(\mathbf{x}^k - \mathbf{x}_0) - \dot{\lambda}_0(\lambda^k - \lambda_0) \end{pmatrix}$$

Detection of bifurcation points



1. Direct indicators f(s)

$$det(\Phi_x(s)) = 0$$

$$\dot{\lambda} = 0$$

2. Solve linear stability problem

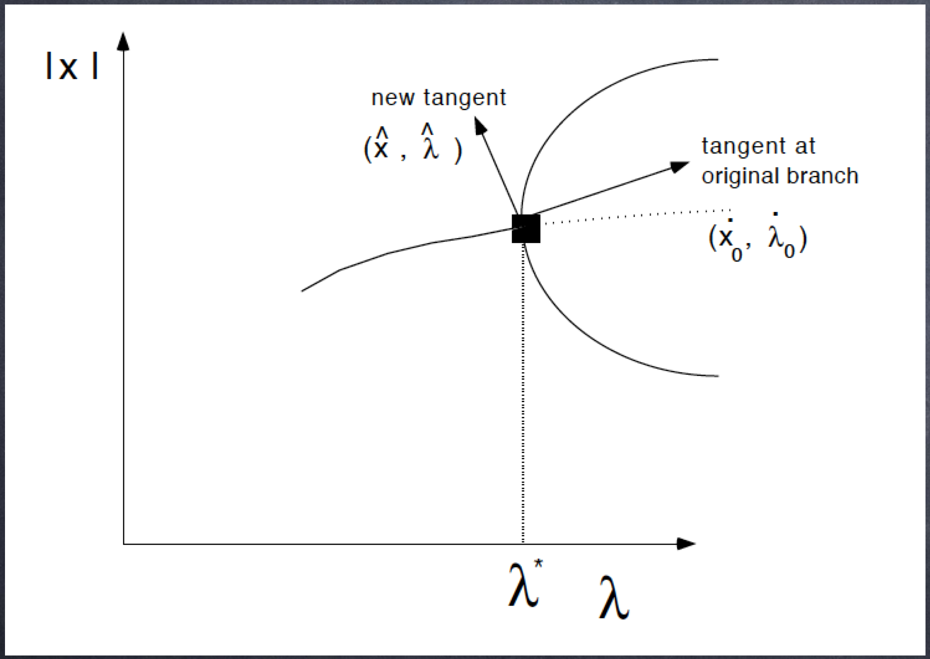
Use secant iteration:

$$s_{l+1} = s_l - f(s_l)\frac{s_l - s_{l-1}}{f(s_l) - f(s_{l-1})}$$

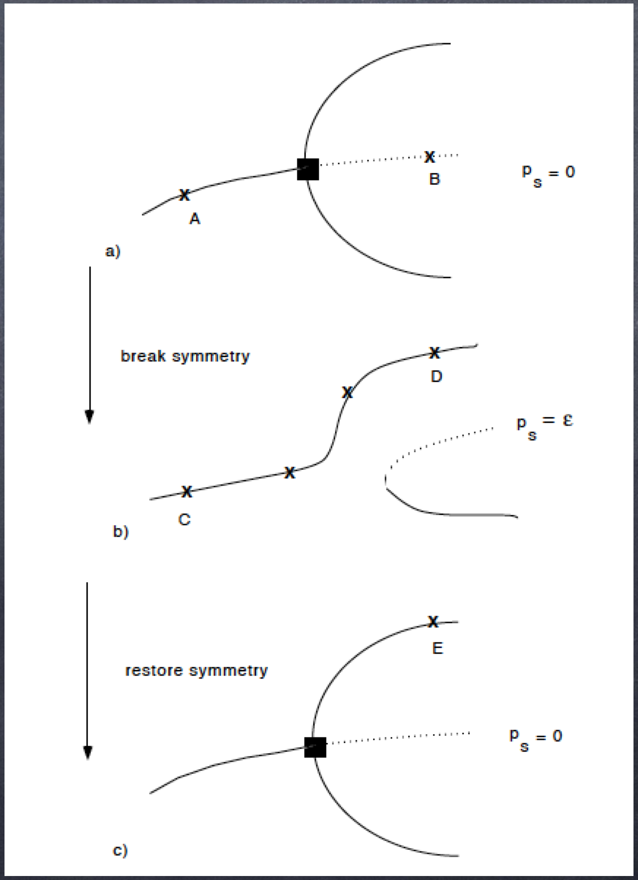
$$s_0 = s_a \quad ; \quad s_1 = s_b$$

Branch switching

1. Orthogonal to tangent



2. Use imperfections



Linear stability

Dynamical system:

$$\mathcal{M}_N \frac{d\mathbf{x}}{dt} + \mathcal{L}_N \mathbf{x} + \mathcal{N}_N(\mathbf{x}) = \mathcal{F}_N$$

The linear stability problem of a fixed point leads to a generalized eigenvalue problem

$$A\mathbf{x} = \sigma B\mathbf{x}$$

Solution methods:

1. QZ
2. Jacobi-Davidson QZ
3. Arnoldi
4. Simultaneous Iteration

Linear stability

$$A\mathbf{x} = \sigma B\mathbf{x}$$

$$\sigma = \sigma_r + i\sigma_i \; ; \; x = \hat{x}_r + i\hat{x}_i$$

How to detect bifurcation points?

Transcritical, Saddle-node, Pitchfork:

A single real eigenvalue crosses the imaginary axis

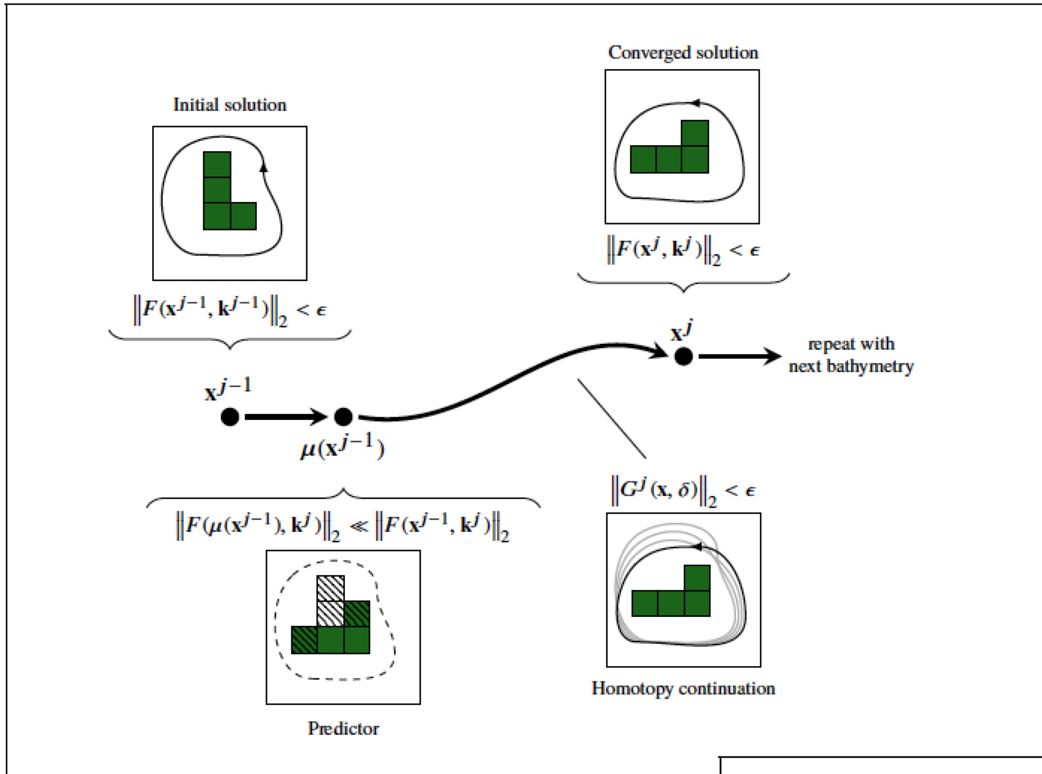
Hopf:

A complex conjugated pair of eigenvalue crosses the imaginary axis

Periodic orbit near Hopf bifurcation?

$$\Phi(t) = e^{\sigma_r t}(\hat{x}_r \cos \sigma_i t - \hat{x}_i \sin \sigma_i t)$$

Extreme continuation: Bathymetry



Mulder et al., Ocean Modeling, (2017).

1: Find $\mathbf{x}^0$ satisfying $\left\|F(\mathbf{x}^0, k^0)\right\|_2 < \epsilon$.

2: **for** $j = 1, 2, \ldots, p-1$ **do**

3: Compute predictor $\mu(\mathbf{x}^{j-1})$ based on difference $\mathbf{k}^j - \mathbf{k}^{j-1}$.

4: Let $G^j(\mathbf{x}, \delta) = \cos^2\theta \; M\left(\mathbf{x} - \mu(\mathbf{x}^{j-1})\right) + \sin^2\theta \; F(\mathbf{x}, \mathbf{k}^j)$. Perform a pseudo-arclength continuation: $\delta = 0 \rightarrow \delta = 1$.

$\theta = \frac{\pi\delta}{2}$.

5: Store $\mathbf{x}^j$ satisfying $\left\|G^j(\mathbf{x}^j, 1)\right\|_2 = \left\|F(\mathbf{x}^j, \mathbf{k}^j)\right\|_2 < \epsilon$

6: **end for**

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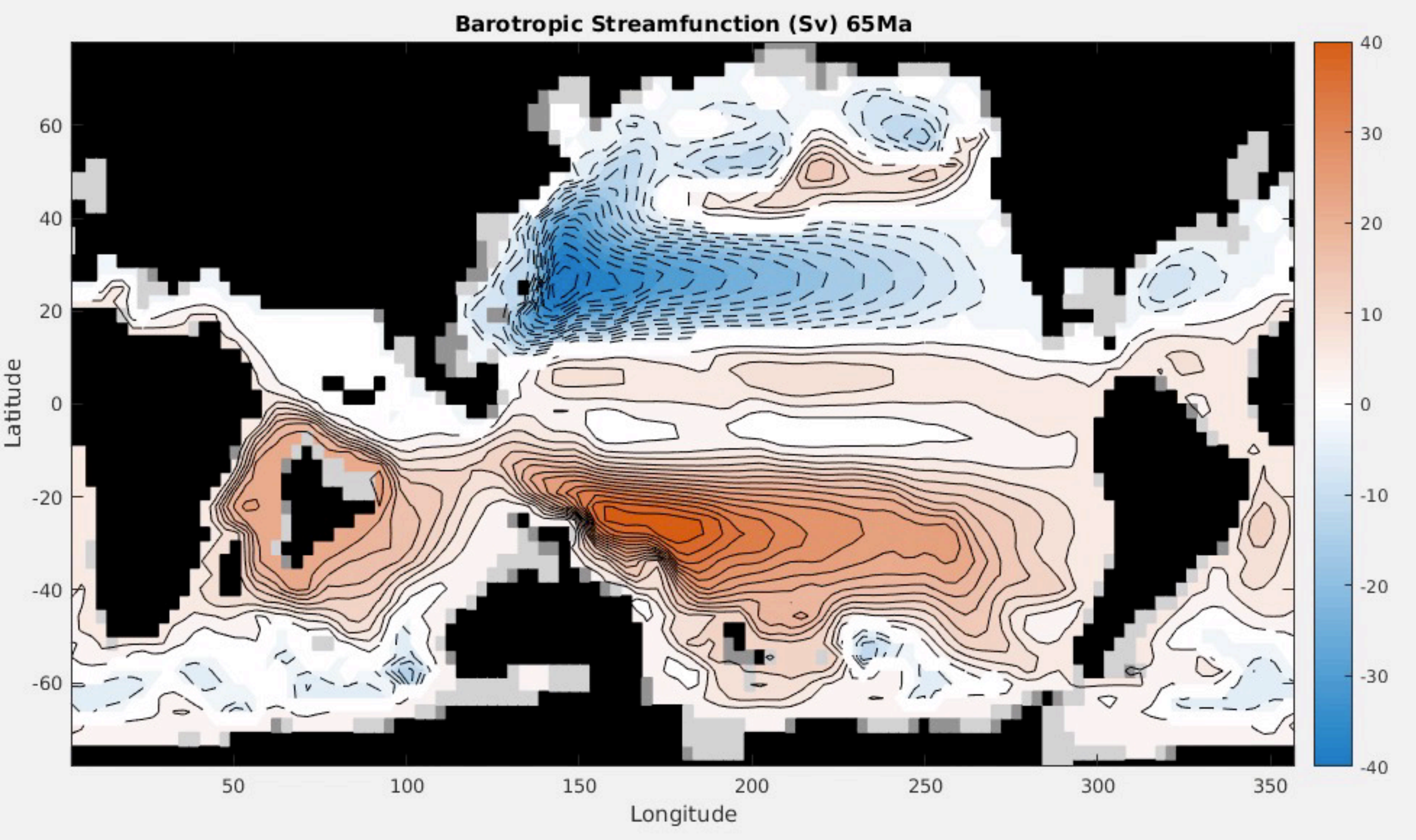
Results (depth averaged circulation), 65M to 0M, idealized forcing



Software

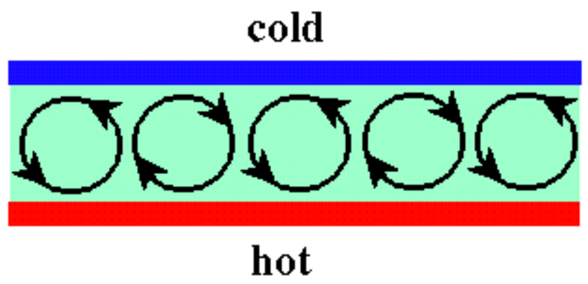
d < 10 : Matcont, Content, PyDSTool, and many more

d < 100 : AUTO

http://indy.cs.concordia.ca/auto/

d < 10^7 : Specialized, tailored codes
(similar setup as AUTO, but with different solvers)

Example: Chaotic system



a ~ vertical temperature difference

Edward Lorenz (1917-2008)

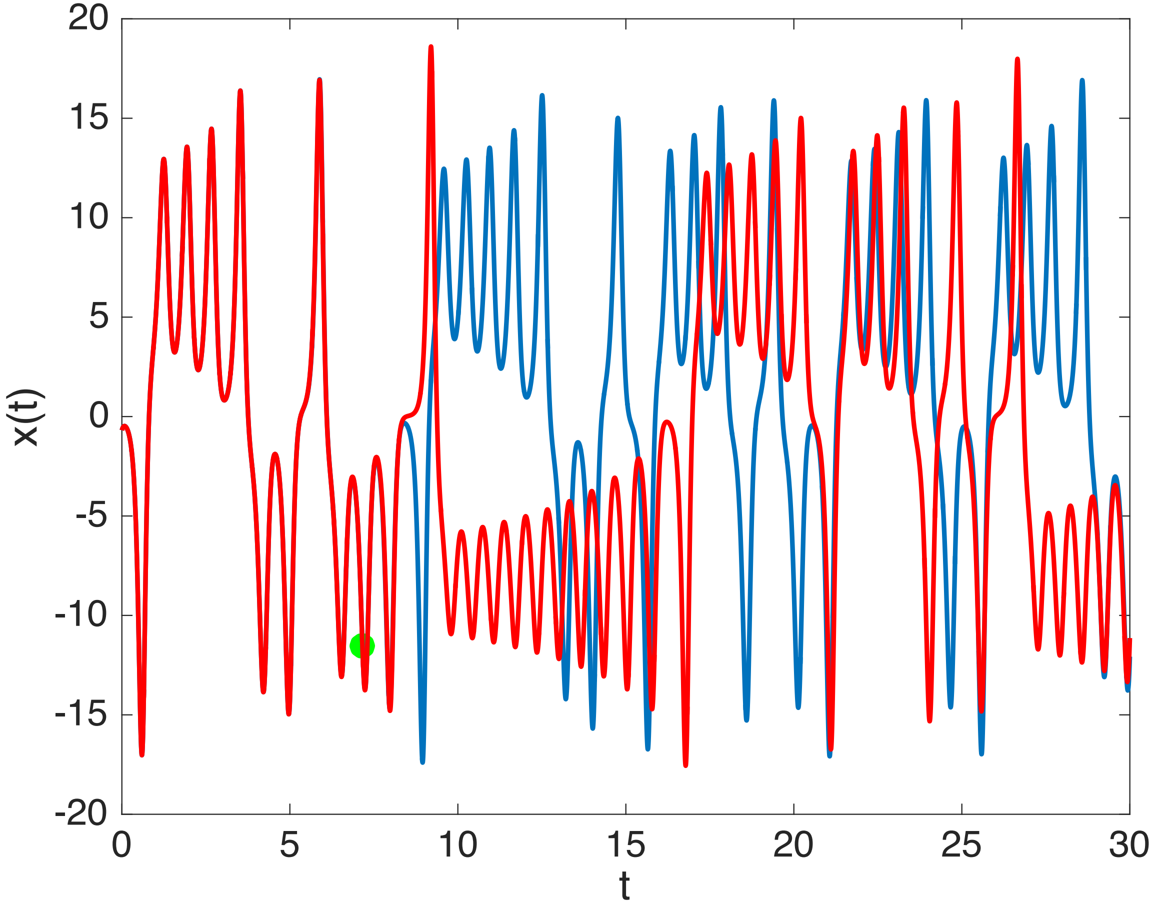
$$\frac{dx}{dt} = -c(x-y)$$

$$\frac{dy}{dt} = ax - y - xz$$

$$\frac{dz}{dt} = b(xy - z)$$

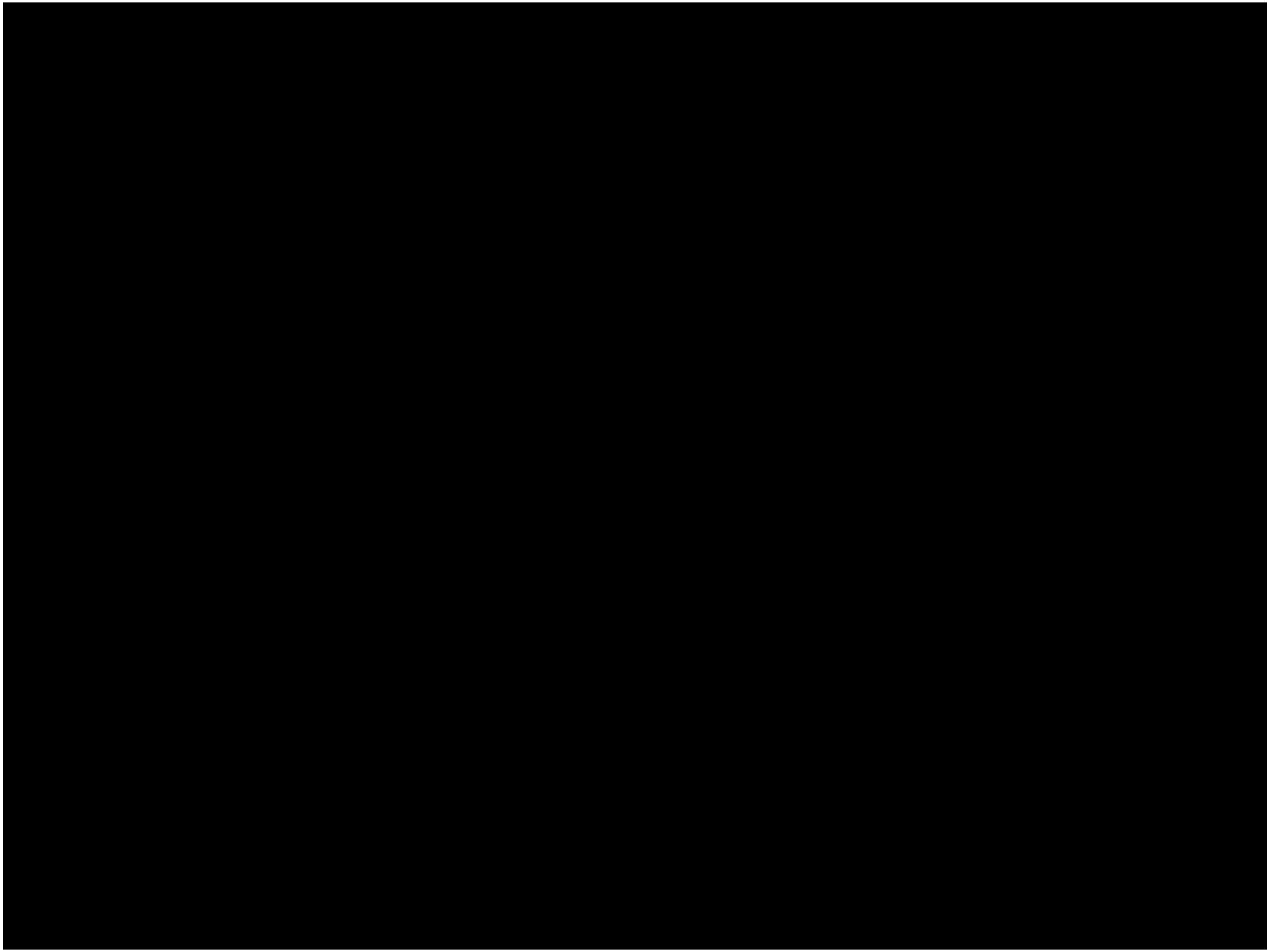
The Lorenz system

Behavior of x(t)

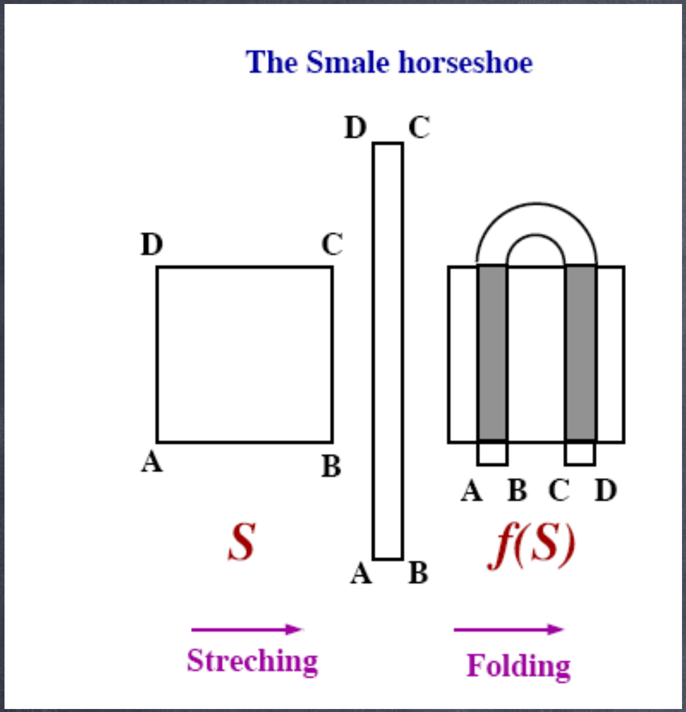


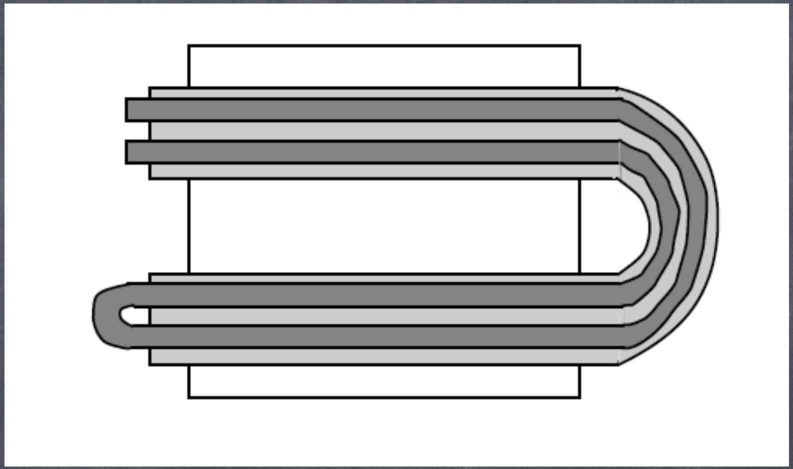
Sensitivity to <u>initial</u> conditions

Spread of trajectories



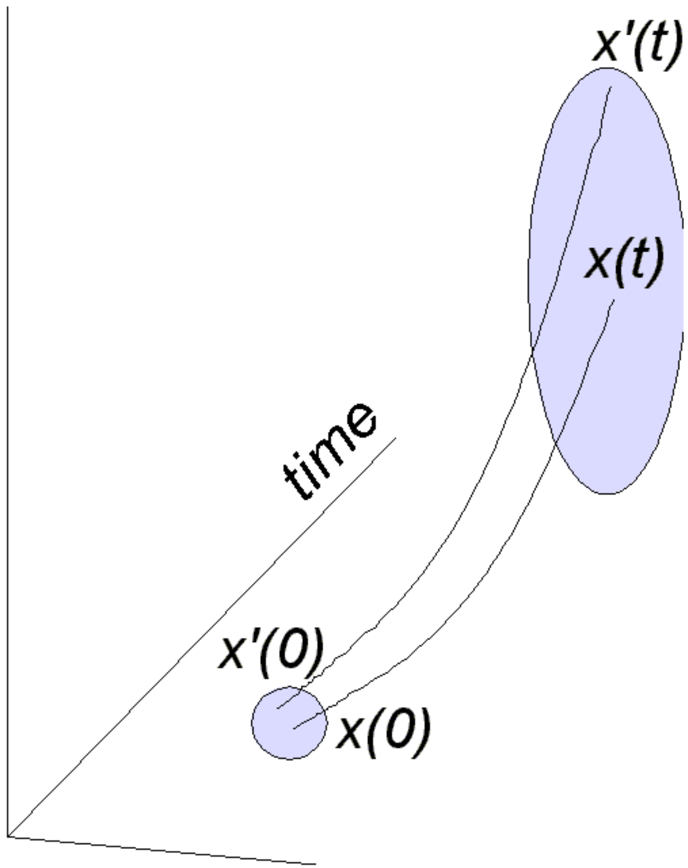
Origin of chaos Horseshoes





Fractal set attractor: Cantor - set

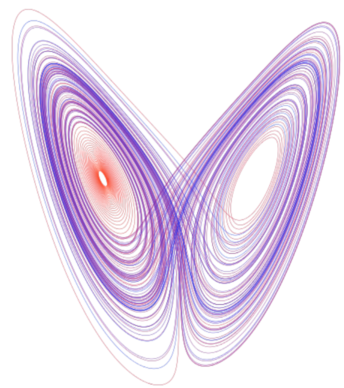
Lyapunov exponent



$$d(t) = x'(t) - x(t)$$

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{d_i(t)}{d_i(0)} \right|$$

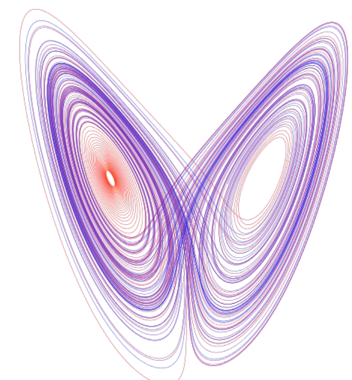
Lorenz system: 0.9056, 0, -14.5723

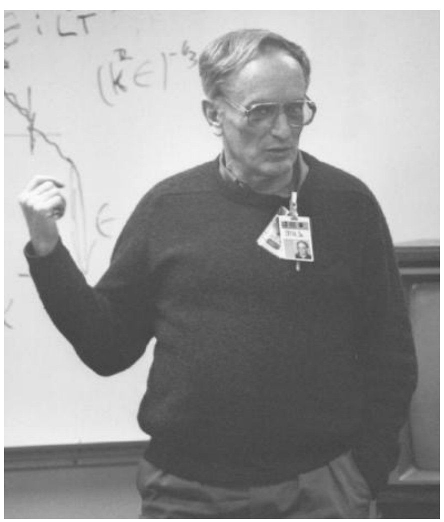


$\lambda > 0 \rightarrow$ chaotic behavior

Weather prediction

Lorenz (1969):
… one flap of a sea-gull's wing may forever change the future course of the weather





Leith (1984):
… even talking about the weather can change the weather!