Date: Tuesday, October 30, 2018.

QUIZ # 2 SOLUTIONS

2a. NOT stationary because it depends on t even, odd.

2b. $\mathbb{E}[W_t] = \mathbb{E}[X_t Y_t]$
$= \mathbb{E}[X_t]\,\mathbb{E}[Y_t]$ because of independence.

Backshift Operator

$$B Y_t = Y_{t-1}$$

$$\begin{aligned} B^2 Y_t &= B \cdot B Y_t \\ &= B Y_{t-1} \\ &= Y_{t-2} \end{aligned}$$

Generally, $\boxed{B^n Y_t = Y_{t-n} \text{ for } n \in \mathbb{N}.}$

Note that $* B^0 = I$.

$B f(t) = f(t-1)$.

AR(P) Process

The time series $\{Y_t\}$ is called an autoregressive process of order p if:

$$\boxed{Y_t = \phi Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t}$$

where $\{\varepsilon_t\} \sim WN(0, \sigma^2)$ and $\phi_1, \phi_2, \dots, \phi_p$ are constants.

An AR(p) process is not necessarily stationary. We require stationarity conditions on the ϕ's to be satisfied.

(Like a regression, but of a time series on itself).

Using backshift operator notation, we can rewrite this relationship in the following way:

$$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \dots - \phi_p Y_{t-p} = \varepsilon_t$$

$$B^0 Y_t - \phi_1 B^1 Y_t - \phi_2 B^2 Y_t - \dots - \phi_p B^p Y_t = \varepsilon_t$$

$$Y_t (1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) = \varepsilon_t$$

$$\left(1 - \sum_{i=1}^{p} \phi_i B^i\right) Y_t = \varepsilon_t$$

Then, we define:

$$\boxed{\phi^p(z) = 1 - \sum_{i=1}^{p} \phi_i z^i}$$

(zeros of this function will define the stationarity)

as the generating function of the AR(p) process.

With this notation, an AR(p) process can be described as:

$$\boxed{\phi^p(B) Y_t = \varepsilon_t.}$$

(we'll do this in depth next week).

MA(q) Process

A time series $\{Y_t\}$ is called a moving average process of order q if

$$\boxed{Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}}$$

where $\{\varepsilon_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \theta_2, \ldots, \theta_q$ are constants.

(It's called like that for historical reasons, it doesn't mean anything Different that the moving average smoother - where it means its name.)

⚠ $MA(q) = AR(\infty)$ (Theoretically, this is true).

Let's see ... "believe me".

MA(1): $Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$

$$\boxed{\varepsilon_t = Y_t - \theta \varepsilon_{t-1}}$$

$$\begin{aligned} Y_t &= \varepsilon_t + \theta(Y_{t-1} - \theta \varepsilon_{t-2}) \\ &= \varepsilon_t + \theta Y_{t-1} - \theta^2 \varepsilon_{t-2} \\ &= \varepsilon_t + \theta Y_{t-1} - \theta^2 (Y_{t-2} - \theta \varepsilon_{t-3}) \\ &= \varepsilon_t + \theta Y_{t-1} - \theta^2 Y_{t-2} + \theta^3 \varepsilon_{t-3} \ldots \end{aligned}$$

Hand-wavy idea to convince you that this is an infinite AR process -

↓

For it to be useful, the answer should be finite.

For this to be true, we require <u>invertibility conditions</u> to be met.

Using backshift operator, we can rewrite the MA(q) relationship:

$$\begin{aligned} Y_t &= B^0 \varepsilon_t + \theta_1 B \varepsilon_t + \theta_2 B^2 \varepsilon_t + \cdots + \theta_q B^q \varepsilon_t. \\ &= (1 + \theta_1 B + \theta_2 B_2 + \cdots + \theta_q B^q) \varepsilon_t \\ &= \left(1 + \sum_{j=1}^{q} \theta_j B^j\right) \varepsilon_t \end{aligned}$$

We define,

$$\boxed{\theta^q(z) = 1 + \sum_{j=1}^{q} \theta_j z^j}$$

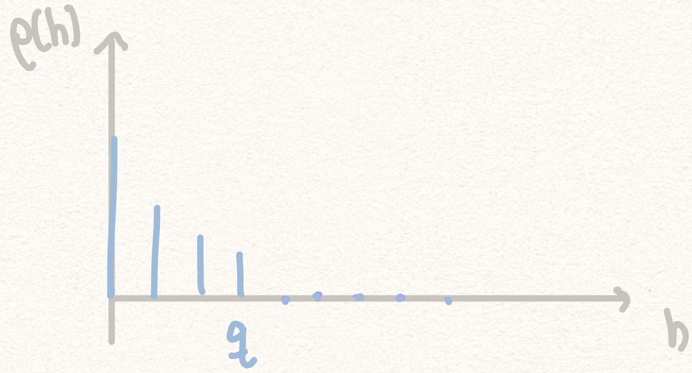
as the generating function of the MA(q) process.

With this notation, we can write MA(q) process as

$$\boxed{Y_t = \theta^q(B)\varepsilon_t.}$$

Note that an MA(q) process is stationary for all q and regardless of the values of the θ's.

An MA(q) process is q-correlated, which means it has non-zero correlation up to lag $h = q$ and zero correlation thereafter.



MA(q) ⟺ q-correlated.

It is also true that <u>any</u> q-correlated process is MA(q).

The practical use for this result is that if we observe real data and the ACF plot behaves in this way, we know an MA(q) process can adequately model this data.

[BREAK] ~ [STARTS AGAIN]

An AR(p) model can be written as an MA(∞) process as long as the stationary conditions are satisfied.

The ACF plot can be used to choose the order q of an MA process but not the order p of an AR process because the ACF of an AR(p) process exhibits exponential decay for all p.

Partial Autocorrelation Function (PACF)

The ACF of lag h measures the correlation between Y_t and Y_{t+h}. This correlation could be due to a direct relationship between Y_t and Y_{t+h}, but it also may be influenced by observations at the intermediate lags:

$$Y_{t+1}, Y_{t+2}, \ldots, Y_{t+h-1}$$

The PACF of lag h measures the correlation between Y_t and Y_{t+h} once the effect of the intermediate lags has been accounted for.

Define

$$\left.\begin{aligned} \hat{Y}_t &= f(Y_{t+1}, Y_{t+2}, \ldots, Y_{t+h-1}) \\ \hat{Y}_{t+h} &= g(Y_{t+1}, Y_{t+2}, \ldots, Y_{t+h-1}) \end{aligned}\right\}$$

where $f(\cdot)$ and $g(\cdot)$ are typically regression $Y_{t+1}, Y_{t+2}, \ldots, Y_{t+h-1}$

For a time series $\{Y_t\}$, the PACF of lag $h \geq 0$ is:

$$\boxed{\alpha(h) = \begin{cases} \operatorname{Corr}(Y_t, Y_t) = 1 & \text{for } h = 0 \\ \operatorname{corr}(Y_t, Y_{t+1}) = \gamma(1) & \text{for } h = 1 \\ \operatorname{corr}(Y_t - \hat{Y}_t, Y_{t+h} - \hat{Y}_{t+h}) & \text{for } h \geq 2 \end{cases}}$$

Example: Derive the PACF for an AR(1) process:

$$y_t = \phi y_{t-1} + \varepsilon_t$$

where $\{\varepsilon_t\} \sim WN(0, \sigma^2)$.

ANSWER:

$$\alpha(h) = \begin{cases} \alpha(0) = 1 & \text{if } h=0 \\ \alpha(1) = \gamma(1) = \phi & \text{if } h=1 \\ \vdots & \end{cases}$$

$\phi 2$

$y_{t-1} = 4$

$y_t = \phi y_{t-1}$

$= 8$

on average, ε's $= 0$

for $h=2$:

$$\begin{aligned} \alpha(2) &= \text{Corr}(y_t - \hat{y}_t,\ y_{t+2} - \hat{y}_{t+2}) \\ &= \text{Corr}(y_t - f(y_{t+1}),\ y_{t+2} - g(y_{t+1})) \\ &= \text{corr}(y_t - f(y_{t+1}),\ y_{t+2} - \phi y_{t+1}) \\ &= \text{corr}(y_t - f(y_{t+1}),\ \varepsilon_{t+2}) \\ &= \underbrace{\text{corr}(y_t, \varepsilon_{t+2})}_{0} - \underbrace{\text{corr}(f(y_{t+1}), \varepsilon_{t+2})}_{0} \\ &= 0 \end{aligned}$$

can't depend on future error

Therefore, for similar reasons,

$$\alpha(h) = \begin{cases} 1 & \text{if } h=0 \\ \phi & \text{if } h=1 \\ 0 & \text{if } h>1 \end{cases}$$

Therefore we can use the PACF plot to identify the order of an AR(p) process.

Remarks :

- If $\{Y_t\} \sim AR(p)$, then the PACF satisfies $\alpha(h) \neq 0$ if $h \leq p$ and $\alpha(h) = 0$ for $h > p$.

- If a PACF exhibits this pattern, then an AR(p) process can model it.

Therefore, we choose p using PACF plots and q using ACF plots.

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