Data: Thursday, October 25, 2018.

Recap

- Strict vs. weak stationarity
- Mean function $\mu(t) = \mathbb{E}[Y_t]$
- Covariance function $\gamma(t, t+h) = \mathrm{Cov}(Y_t, Y_{t+h})$

For a (weak) stationary time series, the covariance function simplifies to:

$\gamma(h)$ does not depend on t.

This function is specifically referred to as an autocovariance function.

In this context, we define the autocorrelation function to be

$$\begin{aligned}
\rho(h) &= \mathrm{Corr}(Y_t, Y_{t+h}), \\
&= \frac{\mathrm{Cov}(Y_t, Y_{t+h})}{\sqrt{\mathrm{Var}(Y_t)\,\mathrm{Var}(Y_{t+h})}}, \\
&= \frac{\mathrm{Cov}(Y_t, Y_{t+h})}{\sqrt{\mathrm{Cov}(Y_t, Y_t)}\sqrt{\mathrm{Cov}(Y_{t+h})}}, \\
&= \frac{\gamma(h)}{\underset{h=0}{\gamma(0)}}.
\end{aligned}$$

PROPERTIES:

1. $\gamma(0) \geq 0 \iff Var[Y_t] \geq 0$
2. $|\gamma(h)| \leq \gamma(0) \iff \frac{|\gamma(h)|}{\gamma(0)} = |\rho(h)| \leq 1$
3. $\gamma(h) = \gamma(-h)$, $\rho(h) = \rho(-h)$

   Even functions

   $Corr(Y_2, Y_5) = Corr(Y_5, Y_2)$

   → we usually want to forecast to the future

Recall that for MA(1) model

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h=0 \\ \sigma^2\theta & \text{if } h=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Noting that $\gamma(0) = \sigma^2(1+\theta^2)$

$$\rho(h) = \begin{cases} 1 & \text{if } h=0 \\ \theta/(1+\theta^2) & \text{if } h=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

You would expect a spike at 1

By looking at an ACF plot, that behaves like that, you could modeling using MA(1).

Example:

First order autoregression AR(1)

A times series $\{Y_t\} \sim AR(1)$ behaves according to the following relationship:

$$Y_t = \phi Y_{t-1} + \varepsilon_t$$

where $|\phi| < 1$ and $\{\varepsilon_t\} \sim WN(0, \sigma^2)$

(?) and ε_t and Y_s are uncorrelated for $s < t$.

This makes AR(1) stationary

Derive the autocovariance function of h and autocorrelation of h. ($\gamma(h)$ and $\rho(h)$).

- $\mathbb{E}[Y_t] = \mathbb{E}[\phi Y_{t-1} + \varepsilon_t]$

$$= \phi \mathbb{E}[Y_{t-1}] + \mathbb{E}[\varepsilon_t] \quad (\mathbb{E}[\varepsilon_t] = 0)$$

$\therefore \mathbb{E}[Y_t] = \phi \mathbb{E}[Y_{t-1}]$

$\mu = \phi \mu$ because $\{Y_t\}$ is stationary

$\Rightarrow \mu = 0$

$\therefore \mu = 0$ since $\phi \neq 1$.

- $\gamma(h) = \text{Cov}(Y_t, Y_{t+h})$

$$= \mathbb{E}[Y_t Y_{t+h}] - \mathbb{E}[Y_t]\mathbb{E}[Y_{t+h}] \quad (\mathbb{E}[Y_t] = 0,\ \mathbb{E}[Y_{t+h}] = 0)$$

$$= \mathbb{E}[Y_t(\phi Y_{t+h-1} + \varepsilon_{t+h})]$$

$$= \mathbb{E}[\theta Y_{t+h-1} Y_t + \varepsilon_{t+h} Y_t]$$

$$= \theta \mathbb{E}[Y_t Y_{t+h-1}] + \mathbb{E}[Y_t \varepsilon_{t+h}] \quad (\mathbb{E}[Y_t \varepsilon_{t+h}] = 0)$$

$$= \theta \gamma(h-1)$$

$$= \theta^2 \gamma(h-2)$$
$$= \theta^3 \gamma(h-3)$$
$$\vdots$$
$$= \theta^h \gamma(0)$$

- $\gamma(0) = Var(Y_t)$

$$= cov(Y_t, Y_t)$$
$$= \mathbb{E}[Y_t^2] - \cancelto{0}{\mathbb{E}[Y_t]}^2$$
$$= \mathbb{E}[(\phi Y_{t-1} + \varepsilon_t)^2]$$
$$= \mathbb{E}[\phi^2 Y_{t-1}^2 + 2\phi Y_{t-1}\varepsilon_t + \varepsilon_t^2]$$
$$= \phi^2 \mathbb{E}[Y_{t-1}^2] + 2\phi\, \mathbb{E}[\cancelto{0}{Y_{t-1}\varepsilon_t}] + \mathbb{E}[\varepsilon_t^2]$$
$$= \phi^2 \gamma(0) + 0 + \sigma^2$$
$$= \phi^2 \gamma(0) + \sigma^2$$
$$\Rightarrow \gamma(0) = \frac{\sigma^2}{1-\phi^2}$$

forward, backward

Therefore, $\gamma(h) = \dfrac{\phi^{|h|}\sigma^2}{1-\phi^2}$ for $h \in \mathbb{Z}$.

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\gamma(h)}{\gamma(0)} = \phi^{|h|} \text{ for } h \in \mathbb{Z}.$$

Exponentially smaller as h increases.

Whereas we have calculated ACF's from specified models, typically we observe actual data and calculate sample estimates of the quantities.

Given an observed time series $\{Y_t\} = \{Y_1, Y_2, \ldots, Y_n\}$

- Mean function:

$$\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t = \hat{\mu}$$

- Sample autocovariance function:

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=}^{n-|h|}(Y_t - \bar{Y})(Y_{t+|h|} - \bar{Y})$$

bias but required for non-negative

- Sample autocorrelation function:

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

The sample ACF can be used to check for "uncorrelatedness" in a time series. This is achieved by comparing $\hat{\rho}(h)$ values to a threshold, which, if exceeded, indicates significant correlation.

These thresholds rely on the asymptotic distribution $\tilde{\rho}(h)$.

For large n, $\quad \tilde{\hat{\rho}}(h) \sim N(0, \frac{1}{n})$

if the time series is uncorrelated.

The threshold is really a 95% confidence interval for $\rho(h)$ and is given by

$$\pm \frac{1.96}{\sqrt{n}}.$$

Thus $\hat{\rho}(h) \notin \left[-\frac{1.96}{\sqrt{n}}, \frac{1.96}{\sqrt{n}} \right]$ is indicative of significant correlation.