Date: Tuesday, October 23, 2018.

QUIZ SOLUTIONS 1

#1a. General

#2a. Period

#3. Plot #1: Homo or <u>hetero</u>

Plot #3: Not obvious... Sinoudial shape: correlation

Plot #4: Clear correlation

#4. Things that require distribution assumption → F

ASSIGNMENT 1

→ Derivation problems (today).

⇒ We need different models than the classical decompossition one.

⇒ Lay fundation (theory). Then, practical.

Generally speaking, a time series is a stochastic process indexed by time.

↓ sequence of random variables

Specifically, we have a sequence of random variables

$$\{Y_t : t \in \mathbb{N}\}$$

random var

where t is an index of time.

(Think about discrete time series — practical)

$y_t \to$ observation from Y_t.

We want to model/analyse data over time, and to do so, we need time series models.

Such a model specifies the joint distribution of $\{Y_1, Y_2, \ldots, Y_n\}$

$$\mathbb{P}(Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_n \leq y_n) \quad \text{for } y_1, y_2, \ldots, y_n \in \mathbb{R}.$$

↳ Think of a joint CDF.

However, specifying this full distribution is generally not possible in practice since there is not enough information to estimate all of the parameters such a model would rely on.

→ Most information is contained in the lower moments of the model's distribution.

BUT most of a distribution information is contained in the 1st and 2nd moment.

- 1st moment: $\mathbb{E}[Y_t]$ for $t = 1, 2, 3, \ldots$

  (Mean)

- 2nd moment: $\mathbb{E}[Y_t Y_{t+h}]$ for $t = 1, 2, 3, \ldots$, $h = 0, 1, 2, \ldots$

  (Variances + covariances) when $t \neq h$: separation in time

The special case: Multivariate normal distribution is specified entirely by its 1st and 2nd moments.

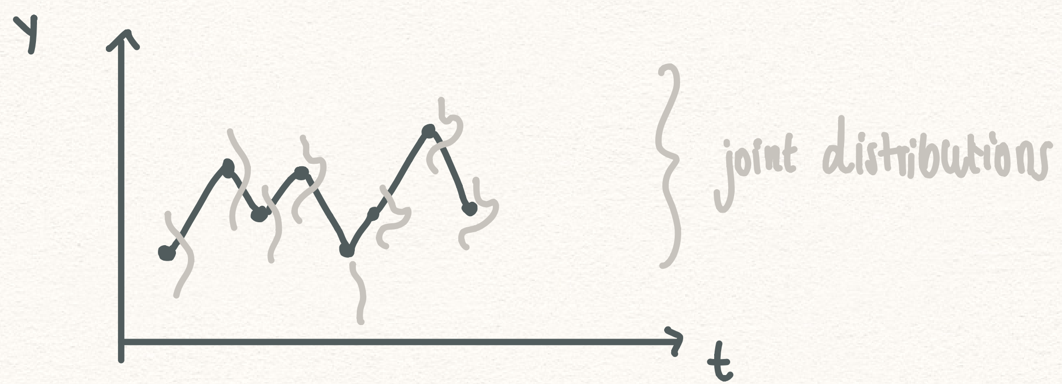
$$\vec{Y} = (Y_1, Y_2, \ldots, Y_n)^T \sim MVN(\vec{\mu}, \Sigma)$$

where

$$\mu_i = \mathbb{E}[Y_i]$$

$$\Sigma_{ii} = Var(Y_i)$$

$$\Sigma_{ij} = Cov(Y_i, Y_j)$$



Take away: We don't need the whole joint distribution, our modeling will be based on 2nd order properties.

There will be some loss of information, but it won't be substancial.

(No loss in the special case).

Zero-mean model ⇒ overly simplistic

- IDD noise, where IDD = identically and independently distributed

$$\{\varepsilon_t\} \sim IDD(0, \sigma^2)$$

set 1,…,n

It could be any distribution. (more general than normal).

→ $\mathbb{E}[\varepsilon_t] = 0 \quad \forall t.$

→ $Var(\varepsilon_t) = \sigma^2 < \infty \quad \forall t.$

→ ε_t and ε_{t+h} are independent for $h>0$. (independent)

→ Every ε_t is identically distributed.

White noise

radio (?) → same variance
more general, more relaxed

$$\{\varepsilon_t\} \sim WN(0, \sigma^2)$$

→ $\mathbb{E}[\varepsilon_t] = 0 \quad \forall t.$ (zero mean)

→ $Var(\varepsilon_t) = \sigma^2 < \infty \quad \forall t.$ (constant variance)

→ $Cov(\varepsilon_t, \varepsilon_{t+h}) = 0$ for $h>0$. (uncorrelated)

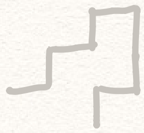
→ Every ε_t is identically distributed.

Different from independent because of $\Leftarrow$ (except normal distribution)

IID noise is a special case of WN.

uncorr $\not\Rightarrow$ indep

Random walk



$$\varepsilon_t = \begin{cases} 1 & \text{w/ probability } 1/2 \\ -1 & \text{w/ probability } 1/2 \end{cases} \Big\} \text{ independently}$$

Bernoulli

$$S_t = \sum_{k=1}^{t} \varepsilon_t$$

and $S_0 = 0$.

- $\mathbb{E}[S_t] = \sum_{k=1}^{t} \mathbb{E}[\varepsilon_t] = \sum_{k=1}^{t} [(1)(1/2) + (0)(1/2)] = 0.$

Once you account for season & trend, account for correlation in error terms.

Stationary: Basic fundamental bubble
↳ we hope is the left over from season & trend
consistent, expected

Strictly Stationary

A time series $\{Y_t\}$ is said to be strictly stationary if the joint distribution of $Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}$ is the same as that of

$$Y_{t_1+h}, Y_{t_2+h}, \ldots, Y_{t_n+h}$$

In other words, $\{Y_t\}$ is strictly stationary if all of its statistical properties remain the same under time shifts.

→ Difficult to verify
→ Never satisfied
→ Require first 2 moments - weak

In practice, this is difficult to verify, and even if you can, it is rarely satisfied. This motivates the need for a weaker definition of stationarity.

But first, we will define the mean function:

$$\mu(t) = \mathbb{E}[Y_t]$$

The covariance function:

$$\gamma(r,s) = \mathrm{Cov}(Y_r, Y_s) = \mathbb{E}[(Y_r - \mathbb{E}[Y_r])(Y_s - \mathbb{E}[Y_s])]$$
$$= \mathbb{E}[Y_r Y_s] - \mathbb{E}[Y_r]\mathbb{E}[Y_s]$$

↳ what happens when independent?

Weak Stationary

use most often
strict ⇒ weak

A time series $\{Y_t\}$ is said to be weakly stationary if

(i) $\mu(t) = \mathbb{E}[Y_t]$ is independent of t.

(ii) $\gamma(t, t+h) =$ is independent of t for all h.
Should look exactly the same, only depend on h. (lag separation).

i.e. covariance only depends on the distance h - not t.

In this scenario $\mu(t) = \mu$, $\gamma(t, t+h) = \gamma(h)$.

- Strict stationary ⇒ weak stationary

If it's not stationary ⇒ different paths.

Example: $\{\varepsilon_t\} \sim \mathrm{IID}(0, \sigma^2)$ — is IID stationary?

- $\mu(t) = \mathbb{E}[\varepsilon_t] = 0$ ✓ Independent of t
- $\gamma(t, t+h) = \mathrm{Cov}(\varepsilon_t, \varepsilon_{t+h}) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h > 0 \end{cases}$ ✓ Independent of t
  ↳ factoring

$\therefore \mathrm{IID}(0, \sigma^2)$ is (weakly) stationary
You could also say it is strict stationary

⇒ Same for white noise. (No factoring but cov still zero).

By similar arguments, you can show that $WN(0, \sigma^2)$ is weakly stationary.

Example: Random walk

$\{S_t\}$ where $S_t = \sum_{k=1}^{t} \varepsilon_k$ and $\{\varepsilon_t\} \sim IID(0, \sigma^2)$

- $\eta(t) = \mathbb{E}[S_t] = \sum_{k=1}^{t} \mathbb{E}[\varepsilon_k] = 0$
- $\gamma(t, t+h) = cov(S_t, S_{t+h})$

$$= cov(S_t, S_t + \varepsilon_{t+1} + \varepsilon_{t+2} + \dots + \varepsilon_{t+h})$$

$$= cov(S_t, S_t) + \cancel{cov(S_t, \varepsilon_{t+1})}^{0} + \dots + \cancel{cov(S_t, \varepsilon_{t+h})}^{0}$$

explode: subscript always different because ε_t are independent

$$= var(S_t)$$

$$= \mathbb{E}[S_t^2] - \cancel{\mathbb{E}[S_t]}^{0}\,{}^2$$

$$= \mathbb{E}\left[\left(\sum_{k=1}^{t} \varepsilon_k\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{t} \varepsilon_k^2 + 2\sum_{i=j} \varepsilon_i \varepsilon_j\right]$$

$$= \sum_{k=1}^{t} \mathbb{E}[\varepsilon_k^2] + 2\sum_{i=j} \mathbb{E}[\varepsilon_i \varepsilon_j]$$

$$= \sum_{k=1}^{t} var[\varepsilon_k] + \sum_{i=j} \cancel{cov(\varepsilon_i, \varepsilon_j)}^{0}$$

0 (indp assump) $\mathbb{E}[\varepsilon_i \varepsilon_j] = \mathbb{E}[\varepsilon_i]\,\mathbb{E}[\varepsilon_j]$

$$= \sum_{k=1}^{t} \sigma^2$$

$$= t\sigma^2 .$$ ✗

(not independent of t) ∴ the random walk is not (weakly) stationary.

Covariance function depends on t
∴ random walk is not weakly stationary.

First order

Example: Moving average model of order one

Notation: MA(1)

Consider the time series $\{Y_t\}$ with model

$$Y_t = \varepsilon_t + \theta \varepsilon_{t-1}$$

where $\{\varepsilon_t\} \sim WN(0, \sigma^2)$.

TASK: Show $\{Y_t\}$ is stationary.

- $\mathbb{E}[Y_t] = \mu(t) = \mathbb{E}[\varepsilon_t + \theta \varepsilon_{t-1}]$
  $= \mathbb{E}[\varepsilon_t] + \theta\, \mathbb{E}[\varepsilon_{t-1}] = 0.$ ✓ indep. of t
- $\mathrm{cov}(Y_t, Y_{t+h}) = \gamma(t, t+h)$

$$= \mathrm{cov}(\varepsilon_t + \theta \varepsilon_{t-1}, \varepsilon_{t+h} + \theta \varepsilon_{t+h-1})$$

$$= \mathrm{cov}(\varepsilon_t, \varepsilon_{t+h}) +$$

$$\theta \quad \mathrm{cov}(\varepsilon_t, \theta \varepsilon_{t+h-1}) +$$

$$\theta \quad \mathrm{cov}(\theta \varepsilon_{t-1}, \varepsilon_{t+h}) +$$

$$\theta^2 \quad \mathrm{cov}(\theta \varepsilon_{t-1}, \theta \varepsilon_{t+h-1})$$

$$= \begin{cases} \sigma^2(1+\theta^2) & h=0 \\ \theta\sigma^2 & h=1 \\ 0 & h \geq 1 \end{cases}$$

✓ indep of t

mismatch

$\therefore$ MA(1) is stationary.

↳ ... $\frac{\mathrm{cov}}{\mathrm{std}}$ autocorr ACF

what is the std of Y?