Date: Tuesday, November 6, 2018

Recap:

- An AR(P) model is stationary iff AR(P) = MA(∞).
  - ↳ This is satisfied iff the zeros of the AR generating function lie outside the unit circle in the complex plane.
  - ↳ This is called the "stationarity condition" for AR models.
- An MA(q) model is only useful if it can be represented as an infinite order AR model (MA(q) = AR(∞)).
  - ↳ This is satisfied iff the zeros of the MA generating function lie outside the unit circle in the complex plane.
  - ↳ This is called the "invertibility condition" for MA models.

ARMA (p, q) Models

$\{Y_t\}$ is an autoregressive moving average process of order p and q if:

$$\boxed{Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t + \theta \varepsilon_{t-1} + \cdots + \theta^q \varepsilon_{t-q}}$$

where $\{\varepsilon_t\} \sim WN(0, \sigma^2)$ and the ϕ's and θ's are constants to be estimated.

The model can also be stated in terms of its generating functions as follows:

$$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p} = \varepsilon_t + \theta \varepsilon_{t-1} + \cdots + \theta^q \varepsilon_{t-q}$$

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) Y_t = (1 + \theta B + \cdots + \theta^q B^q) \varepsilon_t$$

$$\boxed{\phi(B) Y_t = \theta(B) \varepsilon_t}$$

where $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ is the generating function for the AR component,

and $\theta(z) = 1 + \theta_1 z + \dots + \theta_p z^p$ is the generating function for the MA component.

Remarks:

- An ARMA(p,q) model is not necessarily stationary, but we'd like it to be so that we can make model stationarity time series.
- An ARMA(p,q) model is not necessarily invertible, but we'd like it to be so that Y_t can be written exclusively as a function of its own history.
- An ARMA(p,q) is stationary iff its AR component is stationary.

  ↳ we check this by determining whether the AR generating function $\phi(z)$ satisfies the stationarity conditions.
- An ARMA(p,q) is invertible iff its MA component is invertible.

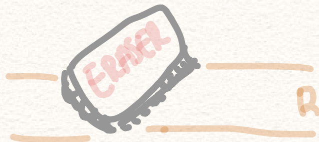
  ↳ we check this by determining whether the MA generating function $\theta(z)$ satisfies the invertibility conditions.

A few more comments:

- $AR(p) = ARMA(p,0)$
- $MA(q) = ARMA(0,q)$
- Model selection is determined by selecting appropriate orders p and q.

  ↳ we can use ACF and PACF plots to help with this. What we expect to see on these plots is the following:

  - ACF: $\underbrace{q}_{h \le q}$ significant spikes + exponential decay.
  - PACF: $\underbrace{p}_{h \le p}$ significant spikes + exponential decay.



Rescued 4!

Not factorial

Example: $\{Y_t\} \sim ARMA(1,2)$

$$Y_t = \phi_1 Y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

Using backshift operator notation, find the generating functions and express this relationship in terms of them.

Solution:

$$Y_t - \phi_1 Y_{t-1} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

$$Y_t(1 - \phi_1 B) = \varepsilon_t (1 + \theta_1 B + \theta_2 B)$$

$$\phi B \, Y_t = \theta(B) \, Y_t$$

where $\phi(z) = 1 - \phi_1 z$

$\theta(z) = 1 + \theta_1 z + \theta_2 z^2$

[BREAK]

Example: $\{Y_t\} \sim ARMA(2,2)$

$$Y_t = Y_{t-1} + 0.5 Y_{t-2} + \varepsilon_t + 0.2 \varepsilon_{t-1} + 0.7 \varepsilon_{t-2}$$

Is $\{Y_t\}$ stationary and/or invertible?

Solution:

$$Y_t - Y_{t-1} - 0.5 Y_{t-2} = \varepsilon_t + 0.2 \varepsilon_{t-1} + 0.7 \varepsilon_{t-2}$$

$$(1 - B - 0.5B^2) Y_t = (1 + 0.2B + 0.7B^2) \varepsilon_t$$

$$\phi B \, Y_t = \theta(B) \, Y_t$$

where $\phi(z) = 1 - z - 0.5z^2$

$\theta(z) = 1 + 0.2z + 0.7z^2$

* $\phi(z) = 0$ iff $z = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(-0.5)(1)}}{2(-0.5)}$

$$= -1 \pm \sqrt{3}$$

$$z_1 = -2.73 \quad \text{and} \quad z_2 = 0.73$$
$$|z_1| = 2.73 \quad \text{and} \quad |z_2| = 0.73$$

$\therefore$ z_2 lies inside the unit circle and so the ARMA(2,2) model is not stationary.

(AR side of things)

* $\theta(z) = 0$ iff $z = \dfrac{-(0.2) \pm \sqrt{(0.2)^2 - 4(0.7)(1)}}{2(0.7)}$

$$= \frac{-0.2 \pm \sqrt{-2.76}}{1.4}$$

$$= \frac{-0.2 \pm \sqrt{2.76}\, i}{1.4}$$

$$z_1 = -0.14 - 1.19i \quad \text{and} \quad z_2 = -0.14 + 1.19i$$

$$|z_1| = \sqrt{(-0.14)^2 + (-1.19)^2} = |z_2| = 1.198$$

Thus $|z_1| = |z_2| > 1$, therefore the zeros lie outside the unit circle and this ARMA(2,2) model is invertible.

$\therefore$ This ARMA(2,2) model is invertible but not stationary.

[R]

Estimating ARMA(p,q) Models

GOAL: To estimate $\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q, \sigma$ in the general ARMA model:

$$\phi(B) Y_t = \theta(B) Y_t$$

(For our purposes, we're going to assume $\mu = 0$.
Why? Because IRL, we usually have to do an operation to make our data from non-stationary to stationary, and in doing so, we mean correct it to zero.
More on that on Tuesday).

These parameters are estimated with observed data $\{Y_1, Y_2, \ldots, Y_n\}$

Several methods of estimation exist. We'll focus on:

(1) Maximum Likelihood Estimation

(2) Least squares estimation

(It won't be as nice as LR because everything is r.v)

ML Estimation

- To do this, we need to make an assumption about the distribution of our data, and hence, the WN process.

↳ $\underset{n\times 1}{\vec{\varepsilon}} \sim MVN(\vec{0}, \sigma^2 \underset{n\times n}{I_n})$ ↳ # of observations

↳ $\underset{n\times 1}{\vec{Y}} \sim MVN(\vec{0}, \underset{n\times n}{\Gamma})$

where
$$\Gamma_{n\times n} = \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(n-1) \\ & \gamma(0) & \gamma(1) & \cdots & \gamma(n-2) \\ & & \gamma(0) & & \vdots \\ & \text{Sym.} & & \ddots & \vdots \\ & & & & \gamma(0) \end{bmatrix}$$

This results in the following likelihood function:

$$L(\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q, \sigma) = \frac{1}{(2\pi)^{n/2}\,|\underbrace{\Gamma_{n\times n}}_{\text{determinant}}|^{1/2}}\, e^{-\frac{1}{2}\vec{y} + \Gamma_{n\times n}^{-1}\vec{y}}$$

where $\vec{y} = (y_1, y_2, \ldots, y_n)^T$.

We want to maximize this function to obtain ML estimates of the parameters.

(It will be done numerically).