Date: Thursday, November 1st. 2018.

Recap

- $\{Y_t\} \sim AR(p)$ if $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + \varepsilon_t$

  $\Rightarrow \phi(B) Y_t = \varepsilon_t$

  where $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$

- $\{Y_t\} \sim MA(q)$ if $Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}$

  $\Rightarrow Y_t = \theta(B) \varepsilon_t$

  where $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$

- $\phi(z)$ and $\theta(z)$ are called "generating functions" aka "characteristic polynomials".

  ↳ necesary to prove stationary and invertivility conditions.

Mathematical Prerequisites

- A power series is an infinite sum representation of a function.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Example:

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ Exponential series

2. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ Geometric series
   Converges for $|x| < 1$

3. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$ Taylor's theorem

- Complex numbers

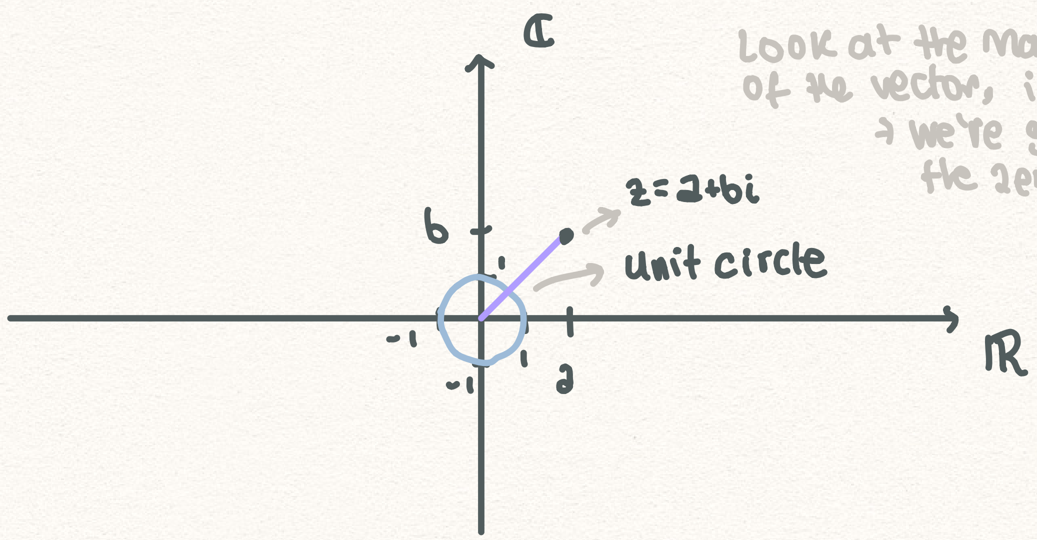
Imaginary number: $\sqrt{-1} \equiv i$.

A complex number can be represented generally as:

$$z = a + bi \in \mathbb{C}$$

where $a, b \in \mathbb{R}$.

Real part　Complex part



Look at the magnitude of the vector, in/out circle.
→ we're going to check the zeros of the polynomial

$|z|$: modulus (magnitude)

$= \sqrt{a^2 + b^2}$

If $|z| > 1$, then z lies outside the unit circle.

If $|z| \leq 1$, then z lies on or inside the unit circle.

Remarks:

- MA(q) is stationary for all q.
- AR(p) = MA(∞), if "stationary conditions" hold.
- MA(q) = AR(p), if "invertibility conditions" hold.

Stationarity conditions

Goal: AR(p) → MA(∞)

$$\phi(B)\, Y_t = \varepsilon_t$$

$$Y_t = \frac{1}{\phi(B)}\, \varepsilon_t \quad (1)$$

Since any function can be written as a power series, let's do that for $\frac{1}{\phi(B)}$:

$$\frac{1}{\phi(B)} = \sum_{n=0}^{\infty} \psi_n B^n \equiv \psi(B)$$

$$= \psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots$$

Plugging this back into (1) yields:

$$Y_t = \psi(B)\, \varepsilon_t$$

$$= (\psi_0 + \psi_1 B + \psi_2 B^2 + \dots)\, \varepsilon_t$$

$$= \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots$$

If $\psi_0 = 1$, then we see that Y_t looks like an MA(∞) process.

Fire drill!

For $\psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots$ to converge, we require that the zeros of $\phi(z)$ lie strictly outside the unit circle in the complex plane.

Thus, an AR(p) model is stationary if the zeros of its generating function lie outside the unit circle in complex plane:

$\phi(z) \neq 0$ for any z such that $|z| \leq 1$

Equivalently,

$\phi(z) = 0$ only for z such that $|z| > 1$

AR Stationarity condition

Invertibility condition

Goal: $MA(q) \to AR(\infty)$

$$Y_t = \theta(B) \varepsilon_t$$

$$\frac{1}{\theta(B)} Y_t = \varepsilon_t \qquad (2)$$

Using the fact that any function has a power series representation, we can write

$$\frac{1}{\theta(B)} = \sum_{n=0}^{\infty} \lambda_n B^n \equiv \lambda(B)$$

$$= \lambda_0 + \lambda_1 B + \lambda_2 B^2 + \dots$$

Plugging this back into (2) yields:

$$\lambda(B) Y_t = \varepsilon_t$$

$$(\lambda_0 + \lambda_1 B + \lambda_2 B^2 + \dots) Y_t = \varepsilon_t$$

$$\lambda_0 Y_t + \lambda_1 Y_{t-1} + \lambda_2 Y_{t-2} + \dots = \varepsilon_t$$

This looks, if $\lambda_0 = 1$, like an $AR(\infty)$ model).

For $\lambda_0 Y_t + \lambda_1 Y_{t-1} + \lambda_2 Y_{t-2} + \dots$ to converge, we require that the zeros of $\theta(z)$ lie outside the unit circle in the complex plane.

In this case we say the MA(q) process is "invertible".

$\theta(z) \neq 0$ for any z such that $|z| \leq 1$

Equivalently,

$\theta(z) = 0$ only for z such that $|z| > 1$

MA invertibility condition

Consequences of:

- $AR(p) = MA(\infty)$

  ↳ ACF of an AR(p) process shows exponential decay because the ACF for MA(q) "shuts off" for $h > q$. But here $q = \infty$, and so AR(q) never "shuts off" on an ACF plot.

- $MA(q) = AR(\infty)$

  ↳ PACF of an MA(q) shows exponential decay because a PACF for AR(p) "shuts off" for $h > p$. But here $p = \infty$ and so MA(q) never "shuts off" on a PACF plot.

Example: $\{Y_t\} \sim AR(2)$

with $Y_t = 0.75\,Y_{t-1} - 0.5625\,Y_{t-2} + \varepsilon_t$

Is $\{Y_t\}$ stationary?

SOLN: $Y_t - 0.75Y_{t-1} + 0.5625Y_{t-2} = \varepsilon_t$

$$Y_t - 0.75\,BY_t + 0.5625\,B^2Y_t = \varepsilon_t$$

$$(1 - 0.75B + 0.5625B^2)\,Y_t = \varepsilon_t$$

$$\therefore \ \phi(z) = 1 - 0.75B + 0.5625B^2$$

For what values of z is $\phi(z) = 0$?

$\phi(z) = 0$ if

$$z = \frac{-(-0.75) \pm \sqrt{(-0.75)^2 - 4(1)(0.5625)}}{2(0.5625)}$$

$$= 2\left(\frac{1 \pm \sqrt{-3}}{3}\right)$$

$$= \frac{2 \pm \sqrt{3}\,i}{3}$$

$$\Rightarrow \ z_1 = \frac{2}{3} - \frac{2\sqrt{3}}{3}i \qquad z_2 = \frac{2}{3} + \frac{2\sqrt{3}}{3}i$$

$$\Rightarrow \ |z| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2\sqrt{3}}{3}\right)^2}$$

$$= \frac{4}{3} > 1$$

Both roots lie outside the unit circle.

$\therefore \ \{Y_t\}$ is stationary.

Example: $\{Y_t\} \sim MA(1)$

with $Y_t = \varepsilon_t + 1.25\varepsilon_{t-1}$

Is it invertible?

SOLN: $Y_t = \varepsilon_t + 1.25\varepsilon_{t-1}$

$= \varepsilon_t + 1.25 B\varepsilon_t$

$= (1 + 1.25B)\varepsilon_t$

$\therefore \ \theta(z) = 1 + 1.25z$

For what values of z is $\theta(z) = 0$?

$\theta(z) = 0$ if

$1 + 1.25z = 0$

$$z = -\frac{1}{1.25} = -0.8$$

$\Rightarrow |z| = 0.8 < 1$

$\therefore$ $\{Y_t\}$ is not invertible.