# 18.06, PSET 9 SOLUTIONS

**Problem 1** First note that $(A./v)_{ij} = A_{ij}/v_i$ and $(A'./v')_{ij} = A_{ji}/v_j$, so these matrices are transpose of each other. So we need to verify that for a 2x2 Markov matrix A and the steady state v $A./v$ is a symmetric matrix. Since A is Markov it is of the form

$$A = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$$

for some $0 \leq a, b \leq 1$. We can check that an eigenvector with eigenvalue 1 is $\begin{pmatrix} b \\ 1-a \end{pmatrix}$, which is non-zero as long as A is not the identity. So the steady state is just this eigenvector normalized to have sum 1, so is given by

$$\begin{pmatrix} \frac{b}{1-a+b} \\ \frac{1-a}{1-a+b} \end{pmatrix}$$

And so

$$A./v = \begin{pmatrix} \frac{a}{b}(1-a+b) & 1-a+b \\ 1-a+b & \frac{1-b}{1-a}(1-a+b) \end{pmatrix}$$

So this matrix is clearly symmetric. The remaining case is the case $A = I$, for which all vectors are eigenvectors of eigenvalue 1 and clearly $A./v$ is symmetric in this case for any steady state v.

**Problem 2** Define $D = diagonal(v)$. Using this matrix $A./v = D^{-1}A$ and $A'./v' = A^T D^{-1}$. So if these 2 are equal we get $A^T = D^{-1}AD$ and so $A$ is diagonally similar to $A^T$

**Problem 3** Suppose we have $A^T = D^{-1}AD$, with $D$ a diagonal matrix with positive entries. Let $S = D^{1/2}$ be the diagonal matrix with entries the square roots of the entries of $D$. Then $D = S^2$ and so $S^{-1}AS = SA^TS^{-1} = (S^{-1}AS)^T$ and so A is similar to a symmetric matrix. As seen in class symmetric matrices have real eigenvalues and similar matrices have the same eigenvalues, thus as $A$ is similar to $S^{-1}AS$, a symmetric matrix, we get that $A$ has real eigenvalues

**Problem 4** The matrix M is antisymmetric and orthogonal as can be checked by computing $M^TM = I$. By the same proof as symmetric matrices having real eigenvalues we get that antisymmetric matrices have purely imaginary eigenvalues. More concretely if $x$ is an eigenvector of $M$ with eigenvalue $\lambda$ then $\lambda\bar{x}^Tx = \bar{x}^TMx = -\bar{x}^TM^Tx = -\bar{\lambda}\bar{x}^Tx$, so $\lambda = -\bar{\lambda}$ and so $\lambda$ is purely imaginary.
Orthogonal matrices have eigenvalues of norm 1. Again let $x$ be an eigenvector with eigenvalue $\lambda$ then $\|x\| = \|Mx\| = |\lambda|\|x\|$, so $|\lambda| = 1$.
It follows that the only possible eigenvalues of $M$ are $\pm i$, so as the trace is 0

$M$ has to have the same amount of eigenvalues $i$ and $-i$, so it has both with multiplicity 2.

**Problem 5** To check if these matrices are positive definite we have to check when the upper left determinants are positive.

(1)
$$S = \begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}$$
The top corner determinant is $1 > 0$ and so the only condition is $9 - b^2 > 0$, ie $-3 < b < 3$

(2)
$$S = \begin{pmatrix} 2 & 4 \\ 4 & c \end{pmatrix}$$
The top corner determinant is $2 > 0$ and so the only condition is $2c - 16 > 0$, ie $c > 8$

(3)
$$S = \begin{pmatrix} c & b \\ b & c \end{pmatrix}$$
The top corner determinant gives the condition $c > 0$ and the determinant gives the condition $c^2 - b^2 > 0$. So this comes down to the condition $c > |b|$.

**Problem 6** To check if these matrices are positive definite we have to check when the upper left determinants are positive.

(1)
$$S = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix}$$
The top corner determinant gives the condition $s > 0$, the top 2x2 determinant gives the condition $s^2 - 16 > 0$ and the full determinant gives the condition $s^3 - 48s - 128 = (s+4)^2(s-8) > 0$. These conditions together give $s > 8$.

(2)
$$S = \begin{pmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{pmatrix}$$
The top corner determinant gives the condition $t > 0$, the top 2x2 determinant gives the condition $t^2 - 9 > 0$ and the full determinant gives the condition $t^3 - 25t = t(t+5)(t-5) > 0$. These conditions together give $t > 5$.

**Problem 7**

(1)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

$A$ has rank 1 with both column and row space equal to $< \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} >$, and if $v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, we get $A = vv^T$ and so $A = (\frac{1}{\|v\|}v)\|v\|^2(\frac{1}{\|v\|}v)^T$ is the SVD of A.

(2)

$$B = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

This matrix is symmetric so the singular values are the absolute values of the eigenvalues. This matrix has rank 2, so we have 2 singular values being 0. The sum of the remaining eigenvalues is $tr(B) = 20 = \lambda_+ + \lambda_-$.

We can compute the square of the sum of the eigenvalues by computing the square sum of the entries of A. So we get $440 = \lambda_+^2 + \lambda_-^2$. It follows from this that $-20 = \lambda_+\lambda_-$. So we get the remaining eigenvalues are $\lambda_\pm = 10 \pm \sqrt{120}$ and so the singular values are $\sigma_+ = \lambda_+$ and $\sigma_- = -\lambda_-$.