1) Use eigenvalues to compute a formula for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^k$$

In [14]:
```
A = [2 1 ; 1 2]
eig(A)
```

Out[14]:
```
([1.0, 3.0], [-0.707107 0.707107; 0.707107 0.707107])
```

Answer: First we diagonalize $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$: the characteristic polynomial is $\det(B - \lambda I) = (2-\lambda)^2 - 1 = 3 - 4\lambda + \lambda^2 = (\lambda-1)(\lambda-3)$ so the eigenvalues are 1 and 3. The eigenvector for $\lambda = 1$ is a basis vector for $N(B - I) = N(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})$ which we can take to be $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (note the julia above gives the normalized form). The eigenvector for $\lambda = 3$ is a basis vector for $N(B - 3I) = N(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix})$ which can be taken to be $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we can diagonalize B as

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

and so

$$A = B^k = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 3^k/2 & 3^k/2 \end{bmatrix} = \begin{bmatrix} 1/2 + 3^k/2 & -1/2 + 3^k/2 \\ -1/2 + 3^k/2 & 1/2 + 3^k/2 \end{bmatrix}$$

2) The Tribonacci numbers are defined in analogy to the Fibonacci numbers:
$T_1 = T_2 = 0,\ T_3 = 1,$
$T_n = T_{n-1} + T_{n-2} + T_{n-3}$ (for $n > 3$)

In [3]:
```
# Inefficient but straightforward computation
T(n) = n>3 ? T(n-1)+T(n-2)+T(n-3) : n==3 ? 1 : 0
[T(n) for n=1:15]'
```

Out[3]:
```
1×15 RowVector{Int64,Array{Int64,1}}:
 0  0  1  1  2  4  7  13  24  44  81  149  274  504  927
```

Let $u_k = \begin{pmatrix} T_{k+2} \\ T_{k+1} \\ T_k \end{pmatrix}$. Find a matrix A that relates u_{k+1} to u_k

In [3]:
```
M = [ 1 2 3; 4 5 6; 7 8 9] # Template for a 3x3 matrix
A = [1 1 1 ; 1 0 0 ; 0 1 0]                     # Write the correct numbers
```

Out[3]:
```
3×3 Array{Int64,2}:
 1  1  1
 1  0  0
 0  1  0
```

Verify numerically that the largest eigenvalue of A is

In [4]:
```
ϕ = (1+∛(19+3*√33)+∛(19-3*√33))/3
```

Out[4]:
```
1.8392867552141612
```

and the other two eigenvalues have absolute value less than 1.

In [5]:
```
abs.(eigvals(A))
```

Out[5]:
```
3-element Array{Float64,1}:
 1.83929
 0.737353
 0.737353
```

Explain why T(31)/T(30) should be about ϕ

Answer: $T(k+3)$ is the first entry of the vector $u_{k+1} = A^k u_1$, i.e. $T(k+3) = e_1^T A^k u_1$, and if X is the matrix of eigenvectors and λ_2, λ_3 are the other two eigenvalues, we have $e_1^T A^k u_1 = e_1^T X \begin{bmatrix} \phi & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}^k X^{-1} u_1$. The idea is that if k is large, the (absolute value of the) powers λ_2^k, λ_3^k are negligible, and so this can be approximated by $e_1^T X \begin{bmatrix} \phi & & \\ & 0 & \\ & & 0 \end{bmatrix}^k X^{-1} u_1 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix} \begin{bmatrix} \phi^k (X^{-1}u_1)_1 \\ 0 \\ 0 \end{bmatrix} = x_{11} \phi^k (X^{-1}u_1)_1$ where $(X^{-1}u_1)_1$ is the first entry of the vector $X^{-1}u_1$. So $T(k+3)/T(k+2) \approx \dfrac{x_{11}\phi^k (X^{-1}u_1)_1}{x_{11}\phi^{k-1}(X^{-1}u_1)_1} = \phi$.

In [6]:
```
T(31)/T(30)
```

Out[6]:
```
1.839286755221798
```

Using Julia, expand u_1 in an eigenvector basis obtaining the coefficients c.
(Two of which are complex, and one may have roundoff as an imaginary part.)

In [4]:
```
Λ,X = eig(A) # Λ is a vector of eigenvalues in Julia for efficiency
c = X\[1,0,0]
```

Out[4]:
```
3-element Array{Complex{Float64},1}:
 -0.727262+2.44804e-17im
 -0.123959+0.46185im
 -0.123959-0.46185im
```

In [8]:
```
real(c[1]*X[1,1]*ϕ^15),T(18)
```

Out[8]:
```
(5767.998305699344, 5768)
```

Answer: This is essentially what we showed above, except we now have the notation $c = X^{-1}u_1$. Setting $k = 15$ above, we have $T(18) = x_{11}\phi^{15} c_1$.

A student wishes to approximate the 18th Tribonacci number.
Explain why the above expression is correct, including the role played by c[1], X[1,1], 15, and 18.

In [9]:
```
T(18) - real(c[1]*X[1,1]*ϕ^15)
```
Out[9]:
```
0.0016943006557994522
```

In [10]:
```
2 * real(c[2]*X[1,2]*Λ[2]^15 )
```
Out[10]:
```
0.0016943006593527162
```

The above formula is the exact error to the student's approximation. Explain.

Answer: Using the expression above we have:

$$T(18) = e_1^T X \begin{bmatrix} \phi & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}^{15} X^{-1} u_1 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix} \begin{bmatrix} \phi & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}^{15} c = x_{11}\phi^{15} c_1 + x_{12}\lambda_2^{15} c_2 +$$

The real part of the first term is our approximation, and the error is $x_{12}\lambda_2^{15} c_2 + x_{13}\lambda_3^{15} c_3$. Since $T(18)$ is real (actually, it's an integer), $T(18) - \mathrm{Re}(x_{11}\phi^{15} c_1) = \mathrm{Re}(x_{12}\lambda_2^{15} c_2 + x_{13}\lambda_3^{15} c_3)$. So all that is left is to show that

$$\mathrm{Re}(x_{12}\lambda_2^{15} c_2 + x_{13}\lambda_3^{15} c_3) = 2\mathrm{Re}(x_{12}\lambda_2^{15} c_2)$$

and this follows from the claim that $x_{12}\lambda_2^{15} c_2$ is the complex conjugate of $x_{13}\lambda_3^{15} c_3$. To see that these are complex conjugates, first note that λ_2 is the complex conjugate of λ_3 (because ϕ, λ_2, and λ_3 form the three roots of a cubic polynomial and ϕ is real, so (λ_2, λ_3) is a pair of complex conjugates). x_{12} and x_{13} are real, as they are the entries of the eigenvector corresponding to ϕ, which is real. Finally, you can observe that $c_2 = \bar{c_3}$:

In [5]:
```
c
```
Out[5]:
```
3-element Array{Complex{Float64},1}:
 -0.727262+2.44804e-17im
 -0.123959+0.46185im
 -0.123959-0.46185im
```

3) For a square nxn matrix A, one variation of the singular value decomposition has $A = U\Sigma V^T$, with all matrices square nxn, and Σ is diagonal with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. (The number of 0 singular values is n-r.)

What is the relationship between the product of the eigenvalues and the product of these n singular values of A?

Answer: The product of the eigenvalues of a square matrix A is $\det(A)$. I claim that the product of the singular values is $|\det(A)|$: first note that $\det(A) = \det(U\Sigma V^T) = \det(U)\det(\Sigma)\det(V^T) = \pm\det(\Sigma)$ since U and V are orthogonal matrices (hence have determinant = ± 1). Furthermore, the entries of Σ are all nonnegative, so this forces the sign.

4) If a matrix has eigenvalue 1, must it have singular value 1? If a matrix has eigenvalue 0, must it have singular value 0?

Answer: For 1, the answer is "no"; take, for example, the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. It has eigenvalue 1 because it is upper triangular with 1 and 0 on the diagonal, but 1 is not a singular value:

In [3]:
```
A = [1 1 ; 0 0]
svd(A)
```

Out[3]:
```
([1.0 0.0; 0.0 1.0], [1.41421, 0.0], [0.707107 -0.707107; 0.707107 0.707107])
```

This does work for 0, however: a square matrix has eigenvalue 0 if and only if it is singular (non-invertible), and a matrix is singular if and only if 0 is one of its singular values (in the square case, in the decomposition $A = U\Sigma V^T$, both U and V^T are invertible, so A is invertible if and only if Σ is invertible).

5) Supose rank(A) = n-1 and x is an eigenvector with eigenvalue 0. How might the information in x find itself inside the SVD?

Answer: Eigenvectors corresponding to the eigenvalue 0 are just (nonzero) vectors in $N(A)$. Since $\operatorname{rank}(A) = n - 1, \dim N(A) = 1$ by the rank-nullity theorem. So x spans $N(A)$. It follows from what you showed on pset 4 that the first $n - 1$ columns of U are an orthonormal basis for $C(A)$; since U is orthogonal, the last column must be orthogonal to this, i.e. a basis for $N(A^T)$. Applying this reasoning to $A^T = V\Sigma U^T$, we see that the last column of V is a unit vector that is a basis for $N(A)$, and so it must be $x/\|x\|$.

6) Construct for every n=2,3,... a non-zero matrix A that has all eigenvalues 0, but has (n-1) singular values 1. Is A diagonalizible?

Answer: Use the $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}$$

This has all zero eigenvalues: just read off the diagonal values. The SVD has $U = I, \Sigma =$ the diagonal matrix with diagonal entries $1, \ldots, 1, 0$ and

$$V^T = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix}$$

A is not diagonalizable: it has just one eigenvalue (0) with multiplicity n, so in order for it to be diagonalizable the corresponding eigenspace would have to have dimension n. But $\lambda = 0$ so eigenspace is $N(A)$ and $\operatorname{rank}(A) = n - 1$ implies $\dim N(A) = 1$.

7) Write an expression for $A^T A$ in terms of the svd of A. Use this to relate the singular values of A to the eigenvalues of $A^T A$. Do the same for AA^T.

Answer: Using the fact that $\Sigma^T = \Sigma$ we have

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^2 V^T.$$

Since $V^T = V^{-1}$, this is a diagonalization of A^T: the eigenvalues are the diagonal entries of Σ^2 (namely σ_i^2 for every i) and the corresponding eigenvectors are the columns of V. For the AA^T part, replace A with A^T in the above: $AA^T = U\Sigma^2 U^T$ is a diagonalization of U whose eigenvalues are the σ_i^2's, and eigenvectors are the columns of U.