18.06, PSET 3 SOLUTIONS

Problem 1a) Denote by $\mathbb{Z} = \{... - 1, 0, 1, 2, ...\}$ the set of integers. Consider $X = \{(x_1, x_2) \in \mathbb{R}^2 | x, y \in \mathbb{Z}\}$. Clearly if $n, m \in \mathbb{Z}$ then $n + m, n - m \in \mathbb{Z}$ so $x, y \in X$ then $x + y, x - y \in X$, but $(1, 0) \in X$ and $(0.5, 0) = 0.5(1, 0) \notin X$
b)Denote by $Y = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = 0 \text{ or } x_2 = 0\}$. If $(x_1, x_2) \in Y$ then if $x_1 = 0$ $cx_1 = 0$ and if $x_2 = 0$ $cx_2 = 0$, so $c(x_1, x_2) \in Y$, but $(1, 0), (0, 1) \in Y$ and $(1, 1) = (1, 0) + (0, 1) \notin Y$

Problem 2 Y above is an example of a union of subspaces which is not a subspace. It is a union of $Y_1 = \{(x_1, 0)\}$ and $Y_2 = \{(0, x_2)\}$
Let $X_1, X_2 \subset X$ be 2 subspaces. If $x, y \in X_1$ and $x, y \in X_2$ then as X_1 and X_2 2 subspaces $x + y \in X_1$ and $x + y \in X_2$ and similarly $cx \in X_1$ and $cx \in X_2$, so the intersection of 2 subspaces is again a subspace.

Problem 3 If A is any matrix and B=I then AB=A, so A and AB have the same column space.
Take A=I (say 3x3) and B=0. Then the column space of A is $\mathbb{R}^3$ and the column space of AB=0 is $\{0\}$

Problem 4 Let the matrix be $C = (c_1\, c_2\, c_3\, c_4\, c_5)$ ie with columns c_i. Since $c_5 = -c_1 - c_3$ the column space of C is the same as the column space of $(c_1\, c_2\, c_3\, c_4)$. But since we are told it has 4 pivots we know the column space of C is the full $\mathbb{R}^4$ and so $(c_1\, c_2\, c_3\, c_4)$ has 4 pivots, so column 5 is sure to not have a pivot
Further as the first 4 columns have pivots we have the only free variable is the 5th variable.
By the above there is only 1 special solution and since the column equation is mantained under row operations, we have $(1, 0, 1, 0, 1)$, is a solution, so has to be the only special solution.

Problem 5 A square matrix has the full number of pivots if and only if the column space is full and so if and only if the determinant is non-zero. But a random matrix has a random determinant and so it is almost sure not to be 0 and thus a random matrix has almost sure 3 pivots and is thus almost sure to have reduced echelon form I
The top 3x3 part of the 4x3 matrix is a random 3x3 matrix and so by the above

is almost sure to have 3 pivots and thus it is almost sure to have reduced form $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Problem 6 First swap the first and third rows to get $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Then take a twice the second row away from the first and divide the first by 2 and the second by 3 to get $R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
Doing the same steps for B we get to $(R\,R)$, which is already in echelon form
Doing the same steps for both blocks of C we get $\begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$ taking the first block from the second we get $\begin{bmatrix} R & R \\ 0 & -R \end{bmatrix}$ and adding the second block to the first and multiplying the second block row by -1 we get $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ And after swapping rows we get $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Problem 7 a) If A, B symmetric matrices, then A+B and cA are both symmetric matrices, so this is already a subspace
b) Consider the set X of matrices such that all rows and columns have the same sum. If A, B are in X, say their columns and rows sum a and b, then A+B has all rows and columns sum to a+b and cA has all rows and columns sum to ca. So X is a subspace. Also for a permutation matrix there is exactly one 1 on each row and column and all other 0, so all columns and rows sum to 1, thus all permutation matrices are in X. Let A be in X. We will take away multiples of the permutation matrices from A until we are left with 0. Then we get that the span of the permutation matrices is exactly X. Using permutation matrices with 1's in different entries of the first column we can take away permutation matrices from A to get a matrix of the form $\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$. Further $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

and similarly we can get $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ as a linear combination of permutation matrices. Taking away multiples of these from the previous matrix we get a matrix of the form $\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}$. But since X is a vector space and A and the permutation matrices are in X, so is this matrix. But the first column is 0, so the sum of each column and row are 0. And so we see the remaining *'s have to be 0's, thus X is the span of the permutation matrices
c) Clearly the identity matrix spans the set of diagonal matrices with all entries on the diagonal equal
d) Denote by E_{ij} the matrix with all entries 0 except $a_{ij} = 1$. It is easy to see that $I + E_{ij}$ are non-singular, since if $i \neq j$ then it is lower/upper triangular with 1's down the diagonal and otherwise it is diagonal with non-zero diagonal entries. Thus the space spanned by all non-singular matrices includes E_{ij}, since I is non-sigular. Clearly these span every matrix, so the set of all non-singular matrices span the space of all matrices
e) Clearly the matrices E_{ij} from d) are singular matrices and they span the whole set of 3x3 matrices, so all singular matrices are the set of all matrices
f) If A, B diagonal matrices A+B and cA are both diagonal, so this is already a subspace

Problem 8 First we compute the reduced echelon form. We start with A. Taking the first row from the second we get $\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ and taking away the second row from the third and twice from the first we get $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is the reduced echelon form for A
To find the echelon form of B take away the second row to the first and twice to the third to get $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, dividing the first row by 2 and the second by 4 we get the echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
To find the special solutions for A, we see the free variables are x_2, x_4, x_5 and so the special solutions are $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$ and $(0, 0, -3, 0, 1)$
Similarly for B, the only free variable is x_3 and so the special solution is $(0, -1, 1)$