PSET 10. SOLUTIONS

Problem 1. a) This is identity transformation and so is obviously linear;

b) We have $T(\lambda \cdot \mathbf{v} + \mu \cdot \mathbf{u}) = T((\lambda \cdot v_1 + \mu \cdot u_1, \lambda \cdot v_2 + \mu \cdot u_2)) = (\lambda \cdot v_1 + \mu \cdot u_1, \lambda \cdot v_1 + \mu \cdot u_1) = \lambda \cdot (v_1, v_1) + \mu \cdot (u_1, u_1) = \lambda \cdot T(\mathbf{v}) + \mu \cdot T(\mathbf{u})$, so T is linear;

c) Analogously to b) $T(\lambda \cdot \mathbf{v} + \mu \cdot \mathbf{u}) = \lambda \cdot (0, v_1) + \mu \cdot (0, u_1) = \lambda \cdot T(\mathbf{v}) + \mu \cdot T(\mathbf{u})$, so T is linear;

d) Is not linear, since $T((0,0)) = (0,1)$ is not zero, but $T((0,0)) = T(0 \cdot v)$ for any v and if T is linear $T(0 \cdot v) = 0 \cdot T(v) = 0$ (zero vector), so T is not linear;

e) $T(\lambda \cdot \mathbf{v} + \mu \cdot \mathbf{u}) = T((\lambda \cdot v_1 + \mu \cdot u_1, \lambda \cdot v_2 + \mu \cdot u_2)) = \lambda \cdot v_1 + \mu \cdot u_1 - (\lambda \cdot v_2 + \mu \cdot u_2) = \lambda \cdot (v_1 - v_2) + \mu \cdot (u_1 - u_2) = \lambda \cdot T(\mathbf{v}) + \mu \cdot T(\mathbf{u})$, so T is linear

Problem 2. a) True since $(A^T)^T = A$;

b) True, since A^T is a 0 matrix if and only if A is (you can also see this by applying ^{T} to both sides of the equation $A^T = 0$: $A = (A^T)^T = 0^T = 0$);

c) True. To find a matrix B, such that $B^T = A$ for a given A you can take A^T, then $(A^T)^T = A$ (and so A is in the range of ^{T});

d) Not true. For $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$ we have $A^T = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -A$. Even simplier example is given by $A = 0$.

Problem 3. We need to find the matrix A such that

$$\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} A = \begin{bmatrix} | & | & | \\ \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \lambda \mathbf{v}_3 \\ | & | & | \end{bmatrix}$$

This matrix is unique (since $V = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}$ is invertible) and one can easily see that $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ fits.

Problem 4. Let X be the transformation matrix we are looking for. We have the following equation on X:

$$UX = AV.$$

We also have $A = U\Sigma V^T$ (SVD decomposition), putting this into the above equation we get

$$UX = U\Sigma V^T V = U\Sigma \Longleftrightarrow X = \Sigma$$

So the tranformation matrix for these input and output basises is given by Σ.

Problem 5. In both parts we need to find a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that

$$\begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} A = \begin{bmatrix} | & | \\ T(\mathbf{v}_1) & T(\mathbf{v}_2) \\ | & | \end{bmatrix}$$

In other words the coefficients of A are given by the coefficients of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ in the (unique) expression as a linear combination of the basis vectors $\mathbf{v}_1, \mathbf{v}_2$: $T(\mathbf{v}_1) = a_{11} \cdot \mathbf{v}_1 + a_{21} \cdot \mathbf{v}_2$ and $T(\mathbf{v}_2) = a_{12} \cdot \mathbf{v}_1 + a_{22} \cdot \mathbf{v}_2$.

a) We have $T(\mathbf{v}_1) = 0 = 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$ and $T(\mathbf{v}_2) = 3\mathbf{v}_1 = 3 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, so $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$;

b) From $T(\mathbf{v}_1) = \mathbf{v}_1$ and $T(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1$ we get $T(\mathbf{v}_2) = T((\mathbf{v}_1 + \mathbf{v}_2) - \mathbf{v}_1) = T(\mathbf{v}_1 + \mathbf{v}_2) - T(\mathbf{v}_1) = \mathbf{v}_1 - \mathbf{v}_1 = 0$. So $T(\mathbf{v}_1) = \mathbf{v}_1 = 1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$ and $T(\mathbf{v}_2) = 0 = 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, and we get $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 6. We identify the space of polynomials of degree ≤ 3 with $\mathbb{R}^4$ where the polynomial $f(x) = a_1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3$ corresponds to a vector $v = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$. This way the basis $1, x, x^2, x^3$ corresponds to the standard basis of $\mathbb{R}^4$ and we have $\frac{d}{dx}$ act as

$$\frac{d}{dx}(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$\frac{d}{dx}(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$\frac{d}{dx}(x^2) = 2 \cdot x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$\frac{d}{dx}(x^3) = 3 \cdot x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3$$

So the matrix of the tranformation is given by $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Problem 7. Analogously

$$T(1) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$T(x) = x + 1 = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$T(x^2) = (x+1)^2 = 1 \cdot 1 + 2 \cdot x + 1 \cdot x^2 + 0 \cdot x^3$$
$$T(x^3) = (x+1)^3 = 1 \cdot 1 + 3 \cdot x + 3 \cdot x^2 + 1 \cdot x^3$$

So the matrix of the tranformation is given by $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Problem 8. 1. No. $T(c\mathbf{v}) = \frac{c\mathbf{v}}{||c\mathbf{v}||} = \frac{c}{||c||}T(\mathbf{v})$ which is not equal to $cT(\mathbf{v})$ unless $||c|| = 1$.

2. Yes. We have $c\mathbf{v} = (cv_1, cv_2, cv_3)$, so $T(c\mathbf{v}) = cv_1 + cv_2 + cv_3 = c(v_1 + v_2 + v_3) = cT(\mathbf{v})$.

3. Yes. We have $c\mathbf{v} = (cv_1, cv_2, cv_3)$, so $T(c\mathbf{v}) = (cv_1, 2cv_2, 3cv_3 = c(v_1, 2v_2, 3v_3) = cT(\mathbf{v})$

4. No. Take $v = (1, 0, 0)$, then $T(v) = 1$, but $T(-v) = 0 \neq -1$