18.06, PSET 9 SOLUTIONS

Problem 1 First note that \((A./v)_{ij} = A_{ij}/v_i\) and 
\((A'./v')_{ij} = A_{ji}/v_j\), so these matrices are transpose of each other. So we need to verify that for a 2x2 Markov matrix \(A\) and the steady state \(v\) \(A./v\) is a symmetric matrix. Since \(A\) is Markov it is of the form 
\[
A = \begin{pmatrix}
a & b \\
1-a & 1-b
\end{pmatrix}
\]
for some \(0 \leq a, b \leq 1\). We can check that an eigenvector with eigenvalue 1 is 
\[
\begin{pmatrix}
b \\
1-a
\end{pmatrix},
\]
which is non-zero as long as \(A\) is not the identity. So the steady state is just this eigenvector normalized to have sum 1, so is given by 
\[
\begin{pmatrix}
b \\
1-a+b
\end{pmatrix}
\]
And so 
\[
A./v = \begin{pmatrix}
a(1-a+b) & \frac{1-a+b}{1-a+b} \\
1-a+b & \frac{1-a}{1-a+b}(1-a+b)
\end{pmatrix}
\]
So this matrix is clearly symmetric. The remaining case is the case \(A = I\), for which all vectors are eigenvectors of eigenvalue 1 and clearly \(A./v\) is symmetric in this case for any steady state \(v\).

Problem 2 Define \(D = \text{diagonal}(v)\). Using this matrix \(A./v = D^{-1}A\) and \(A'./v' = A^TD^{-1}\). So if these 2 are equal we get \(A^T = D^{-1}AD\) and so \(A\) is diagonally similar to \(A^T\).

Problem 3 Suppose we have \(A^T = D^{-1}AD\), with \(D\) a diagonal matrix with positive entries. Let \(S = D^{1/2}\) be the diagonal matrix with entries the square roots of the entries of \(D\). Then \(D = S^2\) and so \(S^{-1}AS = SA^TS^{-1} = (S^{-1}AS)^T\) and so \(A\) is similar to a symmetric matrix. As seen in class symmetric matrices have real eigenvalues and similar matrices have the same eigenvalues, thus as \(A\) is similar to \(S^{-1}AS\), a symmetric matrix, we get that \(A\) has real eigenvalues.

Problem 4 The matrix \(M\) is antisymmetric and orthogonal as can be checked by computing \(M^TM = I\). By the same proof as symmetric matrices having real eigenvalues we get that antisymmetric matrices have purely imaginary eigenvalues. More concretely if \(x\) is an eigenvector of \(M\) with eigenvalue \(\lambda\) then \(\lambda \bar{x}^T x = \bar{x}^T Mx = \bar{x}^T M^T x = -\bar{x}^T x\), so \(\lambda = -\bar{\lambda}\) and so \(\lambda\) is purely imaginary. Orthogonal matrices have eigenvalues of norm 1. Again let \(x\) be an eigenvector with eigenvalue \(\lambda\) then \(\|x\| = \|Mx\| = |\lambda|\|x\|\), so \(|\lambda| = 1\). It follows that the only possible eigenvalues of \(M\) are \(\pm i\), so as the trace is 0.
$M$ has to have the same amount of eigenvalues $i$ and $-i$, so it has both with multiplicity 2.

**Problem 5** To check if these matrices are positive definite we have to check when the upper left determinants are positive.

1. \[ S = \begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix} \]
   The top corner determinant is $1 > 0$ and so the only condition is $9 - b^2 > 0$, i.e. $-3 < b < 3$.

2. \[ S = \begin{pmatrix} 2 & 4 \\ 4 & c \end{pmatrix} \]
   The top corner determinant is $2 > 0$ and so the only condition is $2c - 16 > 0$, i.e. $c > 8$.

3. \[ S = \begin{pmatrix} c & b \\ b & c \end{pmatrix} \]
   The top corner determinant gives the condition $c > 0$ and the determinant gives the condition $c^2 - b^2 > 0$. So this comes down to the condition $c > |b|$.

**Problem 6** To check if these matrices are positive definite we have to check when the upper left determinants are positive.

1. \[ S = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix} \]
   The top corner determinant gives the condition $s > 0$, the top $2 \times 2$ determinant gives the condition $s^2 - 16 > 0$ and the full determinant gives the condition $s^3 - 48s - 128 = (s + 4)^2(s - 8) > 0$. These conditions together give $s > 8$.

2. \[ S = \begin{pmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{pmatrix} \]
   The top corner determinant gives the condition $t > 0$, the top $2 \times 2$ determinant gives the condition $t^2 - 9 > 0$ and the full determinant gives the condition $t^3 - 25t = t(t + 5)(t - 5) > 0$. These conditions together give $t > 5$.

**Problem 7**
(1) 
\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 6 & 8 \\
3 & 6 & 9 & 12 \\
4 & 8 & 12 & 16
\end{pmatrix}
\]

A has rank 1 with both column and row space equal to \(< \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} >\), and if 
\[
v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}
\]
we get \( A = vv^T \) and so \( A = (\frac{1}{\|v\|}v)\|v\|^2(\frac{1}{\|v\|}v)^T \) is the SVD of A.

(2) 
\[
B = \begin{pmatrix}
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 \\
5 & 6 & 7 & 8
\end{pmatrix}
\]

This matrix is symmetric so the singular values are the absolute values of the eigenvalues. This matrix has rank 2, so we have 2 singular values being 0. The sum of the remaining eigenvalues is \( tr(B) = 20 = \lambda_+ + \lambda_- \).

We can compute the square of the sum of the eigenvalues by computing the square sum of the entries of A. So we get \( 440 = \lambda_+^2 + \lambda_-^2 \). It follows from this that \(-20 = \lambda_+ \lambda_- \). So we get the remaining eigenvalues are \( \lambda_\pm = 10 \pm \sqrt{120} \) and so the singular values are \( \sigma_+ = \lambda_+ \) and \( \sigma_- = -\lambda_- \).