PSET 10. SOLUTIONS

Problem 1.  a) This is identity transformation and so is obviously linear;

b) We have \( T(\lambda \cdot v + \mu \cdot u) = T((\lambda \cdot v_1 + \mu \cdot u_1, \lambda \cdot v_2 + \mu \cdot u_2)) = (\lambda \cdot v_1 + \mu \cdot u_1, \lambda \cdot v_1 + \mu \cdot u_1) = \lambda \cdot (v_1, v_1) + \mu \cdot (u_1, u_1) = \lambda \cdot T(v) + \mu \cdot T(u), \) so \( T \) is linear;

c) Analogously to b) \( T(\lambda \cdot v + \mu \cdot u) = \lambda \cdot (0, v_1) + \mu \cdot (0, u_1) = \lambda \cdot T(v) + \mu \cdot T(u), \) so \( T \) is linear;

d) Is not linear, since \( T((0,0)) = (0,1) \) is not zero, but \( T((0,0)) = T(0 \cdot v) \) for any \( v \) and if \( T \) is linear \( T(0 \cdot v) = 0 \cdot T(v) = 0 \) (zero vector), so \( T \) is not linear;

e) \( T(\lambda \cdot v + \mu \cdot u) = T((\lambda \cdot v_1 + \mu \cdot u_1, \lambda \cdot v_2 + \mu \cdot u_2)) = \lambda \cdot v_1 + \mu \cdot u_1 - (\lambda \cdot v_2 + \mu \cdot u_2) = \lambda \cdot (v_1 - v_2) + \mu \cdot (u_1 - u_2) = \lambda \cdot T(v) + \mu \cdot T(u), \) so \( T \) is linear.

Problem 2.  a) True since \((AT)^T = A;\)

b) True, since \(A^T\) is a 0 matrix if and only if \( A \) is (you can also see this by applying \( T \) to both sides of the equation \( A^T = 0; \) \( A = (AT)^T = 0^T = 0;\))

c) True. To find a matrix \( B, \) such that \( B^T = A \) for a given \( A \) you can take \( A^T, \) then \((AT)^T = A \) (and so \( A \) is in the range of \( T);\)

d) Not true. For \( A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} \) we have \( A^T = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -A. \) Even simpler example is given by \( A = 0.\)

Problem 3. We need to find the matrix \( A \) such that

\[
\begin{bmatrix}
    v_1 & v_2 & v_3
\end{bmatrix}
\begin{bmatrix}
    \lambda_1 v_1 \\
    \lambda_2 v_2 \\
    \lambda_3 v_3
\end{bmatrix}
= 
\begin{bmatrix}
    v_1 & v_2 & v_3
\end{bmatrix}
\begin{bmatrix}
    \lambda_1 v_1 \\
    \lambda_2 v_2 \\
    \lambda_3 v_3
\end{bmatrix}
\]

This matrix is unique (since \( V = \begin{bmatrix}
    v_1 & v_2 & v_3
\end{bmatrix} \) is invertible) and one can easily see that \( A = \begin{bmatrix}
    \lambda_1 & 0 & 0 \\
    0 & \lambda_2 & 0 \\
    0 & 0 & \lambda_3
\end{bmatrix} \) fits.

Problem 4. Let \( X \) be the transformation matrix we are looking for. We have the following equation on \( X:\)

\[
UX = AV.
\]
We also have \( A = USV^T \) (SVD decomposition), putting this into the above equation we get

\[
UX = USV^TV = U\Sigma \iff X = \Sigma.
\]
So the transformation matrix for these input and output bases is given by \( \Sigma. \)

Problem 5. In both parts we need to find a matrix \( A = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix} \) such that

\[
\begin{bmatrix}
    v_1 & v_2
\end{bmatrix}
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
= 
\begin{bmatrix}
    T(v_1) & T(v_2)
\end{bmatrix}
\]

In other words the coefficients of \( A \) are given by the coefficients of \( T(v_1) \) and \( T(v_2) \) in the (unique) expression as a linear combination of the basis vectors \( v_1, v_2: \)

\[
T(v_1) = a_{11}v_1 + a_{21}v_2 \quad \text{and} \quad T(v_2) = a_{12}v_1 + a_{22}v_2.
\]

a) We have \( T(v_1) = 0 = 0 \cdot v_1 + 0 \cdot v_2 \) and \( T(v_2) = 3v_1 = 3 \cdot v_1 + 0 \cdot v_2, \) so \( A = \begin{bmatrix}
    0 & 3 \\
    0 & 0
\end{bmatrix};\)

b) From \( T(v_1) = v_1 \) and \( T(v_1 + v_2) = v_1 \) we get \( T(v_2) = T(v_1 + v_2) - T(v_1) = T(v_1 + v_2) - T(v_1) = v_1 - v_1 = 0. \) So \( T(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2 \) and \( T(v_2) = 0 = 0 \cdot v_1 + 0 \cdot v_2, \) and we get \( A = \begin{bmatrix}
    1 & 0 \\
    0 & 0
\end{bmatrix}. \)
Problem 6. We identify the space of polynomials of degree \( \leq 3 \) with \( \mathbb{R}^4 \) where the polynomial \( f(x) = a_1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3 \) corresponds to a vector \( v = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \). This way the basis \( 1, x, x^2, x^3 \) corresponds to the standard basis of \( \mathbb{R}^4 \) and we have \( \frac{d}{dx} \) act as

\[
\begin{align*}
\frac{d}{dx}(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
\frac{d}{dx}(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
\frac{d}{dx}(x^2) &= 2 \cdot x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
\frac{d}{dx}(x^3) &= 3 \cdot x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3
\end{align*}
\]

So the matrix of the transformation is given by

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 
\end{bmatrix}
\]

Problem 7. Analogously

\[
\begin{align*}
T(1) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
T(x) &= x + 1 = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\
T(x^2) &= (x + 1)^2 = 1 \cdot 1 + 2 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 \\
T(x^3) &= (x + 1)^3 = 1 \cdot 1 + 3 \cdot x^2 + 3 \cdot x^3 + 1 \cdot x^3
\end{align*}
\]

So the matrix of the transformation is given by

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 
\end{bmatrix}
\]

Problem 8. 1. No. \( T(cv) = \frac{cv}{||cv||} \) which is not equal to \( cT(v) \) unless \( ||c|| = 1 \).

2. Yes. We have \( cv = (cv_1, cv_2, cv_3) \), so \( T(cv) = cv_1 + cv_2 + cv_3 = c(v_1 + v_2 + v_3) = cT(v) \).

3. Yes. We have \( cv = (cv_1, cv_2, cv_3) \), so \( T(cv) = (cv_1, 2cv_2, 3cv_3) = c(v_1, 2v_2, 3v_3) = cT(v) \)

4. No. Take \( v = (1, 0, 0) \), then \( T(v) = 1 \), but \( T(-v) = 0 \neq -1 \)