Application of Dual-Process Theories in Mathematics Education (and vice versa) [1]

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The gap between intuitive and analytical thinking is of fundamental concern for mathematics education research and practice. There are interesting similarities and differences between the intuitive/analytical framework in mathematics education (ME) on the one hand, and the System 1/System 2 (S1/S2) distinction in dual-process theories (DPT) on the other, but these links so far have hardly been noticed by the two communities. (See Leron & Hazzan, in print).

In this paper I discuss the application of DPT in ME, and, conversely, a possible contribution of ME research to DPT.

DPT consists of a large body of empirical and theoretical work that has been largely unknown in the ME community despite its great relevance. This in itself should be of interest for mathematics educators. But the greatest contribution of DPT to ME, in my view, might come from a change of perspective. Even though the issue hasn't been much discussed explicitly, the prevailing image in ME has been one of *continuity*, of analytical thinking as a refinement of intuitive thinking, of mathematical thinking as an elaboration (however fancy) of common sense. In contrast, the image from DPT of two independent entities, with their different roles and different 'personalities', sometimes cooperating but other times competing for control, turns out to be a new powerful tool in interpreting empirical findings in ME and in suggesting new questions and new experiments. DPT also affords vivid metaphoric and anthropomorphic images, that I find (despite the well-known risks and pitfalls) to be potentially productive in mathematics education research and practice.

These general considerations are illustrated below by comparing a typical use of DPT in cognitive psychology (Example 1) with a novel application in ME (Example 2). By highlighting the similarities and differences, I hope to show the mutual contribution to both communities from linking their separate discourses.

Example 1: In his economics Nobel Prize lecture, Daniel Kahneman (2002) opened with the following story:

> **Question:** *A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost?*
>
> Almost everyone reports an initial tendency to answer '10 cents' because the sum \$1.10 separates naturally into \$1 and 10 cents, and 10 cents is about the right magnitude. [Indeed,] many intelligent people yield to this immediate impulse: 50% (47/93) of Princeton students and 56% (164/293) of students at the University of Michigan gave the wrong answer. (p 451; see also Kahneman & Frederick, 2005, pp 273-4)

Kahneman & Frederick (*ibid*) proceed to use DPT to explain that the surface features of the problem trigger S1's fast and automatic response of '10 cents', since the

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expressions ‘one dollar’ and ‘10 cents’ are salient, and since the orders of magnitude are about right. The roughly 50% of students who answer ‘10 cents’ simply accept S1’s response uncritically. For the rest too, S1 jumps immediately with this answer, but then S2 interferes critically and makes the necessary adjustments to give the normative answer (‘5 cents’).

Example 2. (Leron & Hazzan, in print. *See the appendix for the required mathematical background.*)

The task and data:

> The following task was given to 113 computer science majors in a prestigious Israeli university during an abstract algebra course:
>
> | |
> |---|
> | A student wrote in an exam, "Z_3 is a subgroup of Z_6".
In your opinion, is this statement true, partially true, or false?
Please explain your answer. |
>
> An incorrect answer was given by 73 students, 20 of whom invoked Lagrange's theorem in essentially the following manner:[2]
>
> *Z_3 is a subgroup of Z_6 by Lagrange's theorem, because 3 divides 6.*

The interesting feature of this answer is that it looks virtually *beautiful* by S1, but is complete rubbish by S2.[3] It is easy to imagine not only that students should come up with this answer, but also that they would actually be happy with it, convinced that this is what the instructor had in mind.

Dual-process analysis:

As in the bat-and-ball example, we propose that the students' answers reflect a combined S1-S2 failure.

First, the S1 response is invoked by what is most immediately accessible to the students in the situation, which also looks roughly appropriate to the task at hand; this results in the inappropriate invocation of Lagrange’s theorem as described above. Specifically, the students know that using a theorem in such situations is expected; the appearance of the two numbers 3 and 6 as orders of two groups, and the fact that 3 divides 6, immediately and automatically cues Lagrange’s theorem, yielding the answer, “Z_3 is a subgroup of Z_6 by Lagrange's theorem, because 3 divides 6”.

Second, S2 fails in its role as critic of S1, since there is nothing in the task situation to alert its monitoring function. The missing judgment – mainly that Lagrange’s theorem cannot be used to establish the existence of a subgroup but only its absence – clearly requires S2 involvement. Incidentally, it is quite possible that some of the students do have the knowledge to produce the right answer, had they only stopped to think more (that is, invoke S2). The problem is, rather, that they have no reason to suspect that the answer is wrong, thus their S1 reaction escapes the notice of the “permissive System 2”. (Kahneman, 2002, p. 469)

[2] Hazzan & Leron (1996) discuss the data on two more tasks, which shows that this misuse of Lagrange’s theorem is deeper and more persistent than might appear from the data presented here.

[3] First, the students confuse between Lagrange’s theorem (LT) and its *converse*; second, they use the converse of LT which is not even true; third, they use an incorrect *formulation* of the converse; fourth, there is no way LT could even help in this task, because the task asks whether something is a subgroup, whereas LT *assumes* the subgroup relation.

In ME research, such phenomena would traditionally be explained by students' faulty knowledge of the relevant logics or mathematics. The DPT interpretation, which invokes general cognitive mechanisms rather than specific mathematical knowledge, clearly adds a new perspective for ME researchers. Further experimentation is needed to tell apart the students who had appropriate (S2) mathematical knowledge but who nonetheless yielded to the S1 response.

Conclusion.

The foregoing analysis highlights the *similarity* between the bat-and-ball and the Lagrange theorem examples. I'd like to conclude by highlighting the *difference*.

Cognitive psychologists largely study *everyday* cognition. The tasks given to research participants (such as the bat-and-ball, 'Linda', or the card selection tasks) rarely require prior tutoring or deep reflection. These situations *invite* S1 responses. In contrast, Example 2 was carried out in the context of a university group theory course, where the name of the game is abstraction, definitions, theorems and proofs; and the students are not just asked whether "Z_3 is a subgroup of Z_6", but also prompted to "please explain your answer". This context would clearly seem to invite S2 responses. Thus it is a remarkable finding – and a further testimony to the power of S1– that even in such situations, S1 may still 'run the show' and lead to non-normative responses, such as the mindless invocation of Lagrange's theorem.

Appendix: Mathematical background for Example 2.

The entire task is taking place within the group Z_6, consisting of the set {0,1,2,3,4,5} and the operation of addition modulo 6, denoted by $+_6$. Thus, for example, $2 +_6 3 = 5$, $3 +_6 3 = 0$, $3 +_6 4 = 1$, and, in general, $a +_6 b$ is defined as the remainder of the usual sum a + b upon division by 6.

Z_6 is a *group* in the sense that it contains 0 and is *closed* under addition mod 6; that is, if a and b are in Z_6 , then so is $a +_6 b$.[4] Similarly, we define Z_3 to be the group consisting of the set {0,1,2} and the operation $+_3$ of addition modulo 3. A *subgroup* of Z_6 is a subset of {0,1,2,3,4,5} which is in itself a group under the operation defined in Z_6 . For example, it can be checked that the subset {0,2,4} is a subgroup of Z_6, since it contains 0 and is closed under $+_6$.

All the groups in this discussion are *finite*, in the sense that they have a finite number of elements; this number is called the *order* of the group. Thus the order of Z_6 is 6 and the order of Z_3 is 3. Finally, an important theorem of group theory, called *Lagrange's theorem*, states that *if H is a subgroup of Z_6, then the order of H divides 6.*[5] Thus, for example, the order of H cannot be 4 or 5 but 3 is possible, and indeed, we have seen above an example of a subgroup of Z_6 with 3 elements[6]. For what follows, it is relevant to mention that the *converse* of Lagrange's theorem is *not* true in general: It is possible to give an example of a group G of order 12 which does not contain a subgroup of order 6 (for details see Gallian, 1990, Example 13, p. 151).

[4] In the general definition of a group there are more requirements, namely associativity and the existence of "inverses", but in the present context it can be shown that these are automatically satisfied, so we need not worry about them.

[5] We say that an integer k *divides* an integer n if n is a whole multiple of k. For example, 3 divides 6 but 4 doesn't; 6 divides 12 but 5 doesn't.

[6] More generally, Lagrange's theorem applies to any two finite groups H and G: If H is a subgroup of G, then the order of H divides the order of G.

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