Algebra details and supplementary materials for "On Galerkin approximations for the quasigeostrophic equations" (submitted to JPO)

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1 A slightly different derivation of the enstrophy conservation with $\beta \neq 0$

An equivalent approach to derive the enstrophy conservation with non-zero β is to write the meridional PV flux as the divergence of an Eliassen-Palm vector (e.g. Vallis 2006)

$$vq = \boldsymbol{\nabla \cdot E}\,, \tag{1}$$

where

$$\boldsymbol{E} \stackrel{\text{def}}{=} \tfrac{1}{2}\left(v^2 - u^2 - \left(\tfrac{f_0}{N}\right)^2 \vartheta^2\right)\hat{\boldsymbol{\imath}} - uv\hat{\boldsymbol{\jmath}} + v\vartheta\hat{\boldsymbol{k}}\,. \tag{2}$$

Using $\boldsymbol{E}$, the equation for enstrophy density is

$$\partial_t \tfrac{1}{2}q^2 + \mathsf{J}\left(\psi, \tfrac{1}{2}q^2\right) + \beta\boldsymbol{\nabla \cdot E} = 0\,. \tag{3}$$

Integrating over the volume

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \tfrac{1}{2}q^2\,\mathrm{d}V + \beta\int \underbrace{\left[\partial_x\,\psi\vartheta\right]_{z^-}^{z^+}}_{\boldsymbol{E}\cdot\hat{\boldsymbol{n}}}\,\mathrm{d}S = 0\,. \tag{4}$$

Now take the QGPV equation evaluated at $z^\pm$ and cross-multiply with the boundary conditions to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int q^\pm\vartheta^\pm\,\mathrm{d}S + \beta\int \vartheta^\pm v^\pm\,\mathrm{d}S = 0\,. \tag{5}$$

Eliminating the β terms between (4) and (5) we obtain the enstrophy conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int \tfrac{1}{2}q^2\,\mathrm{d}V - \int q^+\vartheta^+ - q^-\vartheta^-\,\mathrm{d}S\right] = 0\,, \tag{6}$$

which corresponds to M (14)-(15).[1]

2 Approximation B: energy and enstrophy nonconservation

The total energy is approximation B is in the form

$$E_{\mathrm{N}}^B \stackrel{\text{def}}{=} E_\phi + E_\sigma + E_{\phi\sigma}\,. \tag{7}$$

[1]Manuscript equations are denoted M #.

The three terms in (7) are

$$E_\phi = \tfrac{1}{h}\int \tfrac{1}{2}\left[\left|\nabla\phi_{\mathrm{N}}^{B}\right|^2 + \left(\tfrac{f_0}{N}\right)^2\left(\partial_z\phi_{\mathrm{N}}^{B}\right)^2\right]\mathrm{d}V = \sum_{n=0}^{\mathrm{N}}\int\tfrac{1}{2}\left[\left|\nabla\breve{\phi}_n\right|^2 + \kappa_n^2\breve{\phi}_n^2\right]\,\mathrm{d}S\,, \tag{8}$$

$$E_\sigma = \int\tfrac{1}{2}\left[|\nabla\sigma|^2 + \left(\tfrac{f_0}{N}\right)^2(\partial_z\sigma)^2\right]\mathrm{d}V, \tag{9}$$

and

$$E_{\phi\sigma} = \tfrac{1}{h}\int\left[\nabla\phi_{\mathrm{N}}^{B}\cdot\nabla\sigma + \left(\tfrac{f_0}{N}\right)^2\partial_z\phi_{\mathrm{N}}^{B}\,\partial_z\sigma\right]\mathrm{d}V = \sum_{n=0}^{\mathrm{N}}\int\breve{\sigma}_n\triangle_n\breve{\phi}_n\,\mathrm{d}S\,. \tag{10}$$

We form the energy equation by deriving evolution equations for each term in (7) separately. First, we multiply the modal equations M (54) by $-\breve{\phi}_n$, integrate over the horizontal surface, and sum on n, to obtain

$$\frac{\mathrm{d}E_\phi}{\mathrm{d}t} = \sum_{n=0}^{\mathrm{N}}\sum_{s=0}^{\mathrm{N}}\int \mathsf{p}_n\mathsf{p}_s\breve{\phi}_n\mathsf{J}\left(\sigma,\triangle_s\breve{\phi}_s\right)\mathrm{d}V + \beta\sum_{n=0}^{\mathrm{N}}\int\breve{\phi}_n\partial_x\breve{\sigma}_n\mathrm{d}S\,. \tag{11}$$

To obtain the evolution of surface contribution to the total energy (9) we differentiate surface inversion relationship M (45) with respect to time, and then multiply by $-\sigma$, integrate over the volume, and combine with the boundary conditions, to obtain

$$\frac{\mathrm{d}E_\sigma}{\mathrm{d}t} = -\sum_{n=0}^{\mathrm{N}}\int\left[\mathsf{p}_n^+\sigma^+\mathsf{J}\left(\phi_n,\vartheta^+\right) - \mathsf{p}_n^-\sigma^-\mathsf{J}\left(\phi_n,\vartheta^-\right)\right]\,\mathrm{d}S\,. \tag{12}$$

To obtain the evolution of the cross-term $E_{\phi\sigma}$ we first form an equation for $\breve{\sigma}_n$ by combining the boundary conditions

$$\begin{aligned}\partial_t\,\breve{\sigma}_n = &-\tfrac{1}{h}\sum_{m=0}^{\mathrm{N}}\left[\mathsf{p}_n^+\mathsf{p}_m^+\triangle_n^{-1}\mathsf{J}\left(\phi_m,\vartheta^+\right) - \mathsf{p}_n^-\mathsf{p}_m^-\triangle_n^{-1}\mathsf{J}\left(\phi_m,\vartheta^-\right)\right]\\ &+\tfrac{1}{h}\mathsf{p}_n^+\triangle_n^{-1}\mathsf{J}\left(\sigma^+,\vartheta^+\right) - \tfrac{1}{h}\mathsf{p}_n^-\triangle_n^{-1}\mathsf{J}\left(\sigma^-,\vartheta^-\right)\,,\end{aligned} \tag{13}$$

where the n'th mode Helmholtz operator is

$$\triangle_n \stackrel{\mathrm{def}}{=} \triangle - \kappa_n^2\,, \tag{14}$$

Now multiply (13) by $\triangle_n\breve{\phi}_n$, integrate over the horizontal surface, and sum over n, to obtain

$$\begin{aligned}\sum_{n=0}^{\mathrm{N}}\int\triangle_n\breve{\phi}_n\partial_t h\,\breve{\sigma}_n\mathrm{d}S = &-\sum_{n=0}^{\mathrm{N}}\sum_{m=0}^{\mathrm{N}}\int\left[\mathsf{p}_n^+\mathsf{p}_m^+\triangle_n\breve{\phi}_n\triangle_n^{-1}\mathsf{J}\left(\breve{\phi}_m,\vartheta^+\right)\right.\\ &\left.-\mathsf{p}_n^-\mathsf{p}_m^-\triangle_n\breve{\phi}_n\triangle_n^{-1}\mathsf{J}\left(\breve{\phi}_m,\vartheta^-\right)\right]\mathrm{d}S + \int\left[\mathsf{p}_n^+\triangle_n\breve{\phi}_n\triangle_n^{-1}\mathsf{J}\left(\sigma^+,\vartheta^+\right)\right.\\ &\left.-\mathsf{p}_n^-\triangle_n\breve{\phi}_n\triangle_n^{-1}\mathsf{J}\left(\sigma^-,\vartheta^-\right)\right]\mathrm{d}S\,.\end{aligned} \tag{15}$$

The linear operator $\triangle_n$ is self-adjoint so that

$$\int\triangle_n^{-1}\mathsf{J}(A,B)\triangle_n C\,\mathrm{d}S = \int C\,\mathsf{J}(A,B)\mathrm{d}S\,. \tag{16}$$

Hence the double sum terms vanish by skew-symmetry of the Jacobian, and we are left with

$$\sum_{n=0}^{\mathrm{N}} \int \triangle_n \breve{\phi}_n \partial_t \, h \breve{\sigma}_n \mathrm{d}S = + \int \left[\mathsf{p}_n^+ \breve{\phi}_n \mathsf{J}\left(\sigma^+, \vartheta^+\right) - \mathsf{p}_n^- \breve{\phi}_n \mathsf{J}\left(\sigma^-, \vartheta^-\right) \right] \mathrm{d}S \,, \tag{17}$$

We then multiply the modal equations M (54) by $-h\,\breve{\sigma}_n$ and add it to $-$(17) to obtain

$$\begin{aligned} \frac{\mathrm{d}E_{\phi\sigma}}{\mathrm{d}t} = & \sum_{n=0}^{\mathrm{N}} \sum_{m=0}^{\mathrm{N}} \sum_{s=0}^{\mathrm{N}} \Xi_{mns} \int \breve{\sigma}_n \mathsf{J}\left(\breve{\phi}_n, \triangle_s \breve{\phi}_s\right) \mathrm{d}S - \int \left[\mathsf{p}_n^+ \breve{\phi}_n \mathsf{J}\left(\sigma^+, \vartheta^+\right) - \mathsf{p}_n^- \breve{\phi}_n \mathsf{J}\left(\sigma^-, \vartheta^-\right) \right] \mathrm{d}S \\ & + \sum_{n=0}^{\mathrm{N}} \sum_{s=0}^{\mathrm{N}} \int \mathsf{p}_n \mathsf{p}_s \breve{\sigma}_n \mathsf{J}\left(\sigma, \triangle_s \breve{\phi}_s\right) \mathrm{d}V + \beta \sum_{n=0}^{\mathrm{N}} \int \breve{\sigma}_n \partial_x \breve{\phi}_n \mathrm{d}S \,. \end{aligned} \tag{18}$$

Adding (11), (12), and (18) we finally obtain the energy equation in approximation B

$$\begin{aligned} \frac{\mathrm{d}E_{\mathrm{N}}^B}{\mathrm{d}t} = & \sum_{n=0}^{\mathrm{N}} \sum_{s=0}^{\mathrm{N}} \int \mathsf{p}_n \mathsf{p}_s \breve{\phi}_n \mathsf{J}(\sigma, \triangle_s \breve{\phi}_s) \mathrm{d}V + \sum_{m=0}^{\mathrm{N}} \sum_{n=0}^{\mathrm{N}} \sum_{s=0}^{\mathrm{N}} \Xi_{mns} \int \breve{\sigma}_n \mathsf{J}(\breve{\phi}_m, \triangle_s \breve{\phi}_s) \, h \mathrm{d}S \\ & + \sum_{n=0}^{\mathrm{N}} \sum_{s=0}^{\mathrm{N}} \int \mathsf{p}_n \mathsf{p}_s \breve{\sigma}_n \mathsf{J}(\sigma, \triangle_s \breve{\phi}_s) \mathrm{d}V \,. \end{aligned} \tag{19}$$

2.0.1 The simplest model

The crudest approximation with non-zero surface buoyancy considers a barotropic interior dynamics ($\mathrm{N} = 0$): $q_{\mathrm{N}}^{\mathrm{G}} = \breve{q}_0$ and $\phi_{\mathrm{N}}^B = \phi_0$. Notice that

$$\int_{z^-}^{z^+} \sigma \, \mathrm{d}z = h \, \breve{\sigma}_0 \,. \tag{20}$$

Hence, the last term on the right-hand-side of (19) vanishes identically. Moreover, the first and second terms on the right-hand-side of (19) cancel out because $\Xi_{000} = 1$. Thus, this simplest model conserves energy. The interior equation is

$$\partial_t \breve{q}_0 + \mathsf{J}\left(\phi_0 + \breve{\sigma}_0 \,, \breve{q}_0\right) + \beta \, \partial_x \left(\phi_0 + \breve{\sigma}_0\right) = 0 \,, \tag{21}$$

where

$$\triangle \breve{\phi}_0 = \breve{q}_0 \qquad \text{and} \qquad \triangle \breve{\sigma}_0 = -\tfrac{1}{h} \left(\vartheta^+ - \vartheta^-\right) \,. \tag{22}$$

The boundary conditions are

$$\partial_t \vartheta^\pm + \mathsf{J}\left(\phi_0 + \sigma^\pm \,, \vartheta^\pm\right) = 0 \,. \tag{23}$$

2.0.2 An example of energy non-conservation

As described in appendix A of Rocha et al., with richer interior dynamics, approximation B does not conserve energy. The simplest example with sheared interior dynamics is $\mathrm{N} = 1$ with constant stratification: $q_{\mathrm{N}}^{\mathrm{G}} = \triangle_0 \breve{\phi}_0 + \sqrt{2}\, \triangle_1 \breve{\phi}_1 \cos(\pi z)$ and $\phi_{\mathrm{N}}^B = \phi_0 + \sqrt{2}\, \phi_1 \cos(\pi z)$. This is the "two surfaces and two modes" model of Tulloch & Smith (2009). The only non-zero entries of the interaction tensor are $\Xi_{000} = \Xi_{011} = \Xi_{101} = \Xi_{110} = 1$. Using this, and noticing that

$$\mathsf{p}_0 \, \mathsf{p}_1 = \mathsf{p}_1 \qquad \text{and} \qquad \mathsf{p}_1 \, \mathsf{p}_1 = \mathsf{p}_0 + \frac{\mathsf{p}_2}{\sqrt{2}} \,, \tag{24}$$

the energy equation (19) becomes, after many cancellations,

$$\frac{\mathrm{d}E_1^B}{\mathrm{d}t} = \int \left[\phi_1 \,\mathsf{J}\left(\breve{\sigma}_1\,,\breve{q}_1\right) - \tfrac{1}{\sqrt{2}}\breve{q}_1 \,\mathsf{J}\left(\breve{\sigma}_1\,,\breve{\sigma}_2\right)\right] \mathrm{d}S\,, \tag{25}$$

where we considered $= h = 1$ for simplicity. The choice made in appendix A of Rocha et al. is $\triangle_1\breve{\phi}_1 = \lambda\phi_1$, where λ is a constant, so that the first term on the right-hand-side of (25) is identically zero. As for the surface streamfunction, we choose

$$\sigma = \frac{\cosh\left(z+1\right)}{\sinh 1}\cos x + \frac{\cosh z}{\sinh 1}\sin x\,. \tag{26}$$

Some useful intermediate results are

$$\breve{\sigma}_1 = \int_{-1}^{0} \mathsf{p}_1\sigma\,\mathrm{d}z = \frac{\sqrt{2}}{1+\pi^2}\left(\cos x - \sin y\right)\,, \tag{27}$$

and

$$\breve{\sigma}_2 = \int_{-1}^{0} \mathsf{p}_2\sigma\,\mathrm{d}z = \frac{\sqrt{2}}{1+4\pi^2}\left(\cos x + \sin y\right)\,. \tag{28}$$

Thus,

$$\mathsf{J}\left(\breve{\sigma}_1\,,\breve{\sigma}_2\right) = \frac{-4}{1+5\pi^2+4\pi^4}\sin x\,\cos y\,. \tag{29}$$

Using these results leads to the energy non-conservation results M (A10).

Enstrophy

To obtain an enstrophy equation for approximation B, we multiply the interior equations M (54) by $\triangle_n\breve{\phi}_n$ and integrate over the surface to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\mathrm{N}}\int\frac{(\triangle_n\breve{\phi}_n)^2}{2}\mathrm{d}S - \beta\sum_{n=0}^{\mathrm{N}}\int\triangle_n\breve{\sigma}_n\partial_x\breve{\phi}_n\mathrm{d}S = 0\,. \tag{30}$$

The enstrophy

$$\sum_{n=0}^{\mathrm{N}}\int\frac{(\triangle_n\breve{\phi}_n)^2}{2}\mathrm{d}S\,, \tag{31}$$

is conserved with $\beta = 0$. For non-zero β, we attempt to obtain a conservation law by eliminating the $\beta-$term. First we form an equation for $\triangle_n\breve{\sigma}_n$ by combining the boundary conditions

$$\partial_t\triangle_n\breve{\sigma}_n - \frac{1}{h}\sum_{m=0}^{\mathrm{N}}\breve{q}_n\left[(\mathsf{p}_n^+\mathsf{J}\left(\sigma^+ + \mathsf{p}_m^+\breve{\phi}_m\,,\vartheta^+\right) - \mathsf{p}_n^-\mathsf{J}\left(\sigma^- + \mathsf{p}_m^-\breve{\phi}_m\,,\vartheta^-\right)\right] = 0\,. \tag{32}$$

We then cross-multiply (32) with the modal equations M (54), integrate over the surface S, and sum on n, the resulting equation with (30) to eliminate β. The final result is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\mathrm{N}}\frac{(\triangle_n\breve{\phi}_n)^2}{2} + (\triangle_n\breve{\sigma}_n)\breve{q}_n\mathrm{d}S = \\
&-\sum_{m=0}^{\mathrm{N}}\sum_{n=0}^{\mathrm{N}}\sum_{s=0}^{\mathrm{N}}\Xi_{mns}\int\triangle_n\breve{\sigma}_n\mathsf{J}\left(\breve{\phi}_m\,,\triangle_s\breve{\phi}_s\right)\mathrm{d}S - \sum_{n=0}^{\mathrm{N}}\tfrac{1}{h}\int\triangle_n\breve{\sigma}_n\mathsf{p}_n\mathsf{p}_n\mathsf{J}\left(\sigma\,,\triangle_s\breve{\phi}_s\right)\mathrm{d}V \\
&+\tfrac{1}{h}\sum_{n=0}^{\mathrm{N}}\sum_{m=0}^{\mathrm{N}}\breve{q}_n\left[\mathsf{p}_n^+\mathsf{J}\left(\sigma^+ + \mathsf{p}_m^+\breve{\phi}_m\,,\vartheta^+\right) - \mathsf{p}_n^-\mathsf{J}\left(\sigma^- + \mathsf{p}_m^-\breve{\phi}_m\,,\vartheta^-\right)\right]\,.
\end{aligned} \tag{33}$$

2.0.3 The simplest model

For the most crude truncation (33) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{(\triangle_0\breve{\phi}_0)^2}{2} - \tfrac{1}{h}\left(\vartheta^+ - \vartheta^-\right)\triangle_0\breve{\phi}_0 \mathrm{d}S = \\ -\int \breve{q}_0 \mathsf{J}\left(\breve{\sigma}_0\,, \vartheta^+ - \vartheta^-\right)\mathrm{d}S + \int \triangle_0\breve{\phi}_0\Big[\mathsf{J}\left(\sigma^+\,, \vartheta^+\right) - \mathsf{J}\left(\sigma^-\,, \vartheta^-\right)\Big]\mathrm{d}S\,. \tag{34}$$

The right-hand-side of (34) is generally non-zero. We can always choose an initial condition for which enstrophy is guaranteed to grow or decay. Choosing

$$\sigma = \frac{\cosh\left[z+1\right]}{\sinh 1}\cos x + \frac{\cosh\left[2(z+1)\right]}{\sinh 2}\cos 2y\,, \tag{35}$$

where obtain

$$\vartheta^+ = \cos x + 2\cos 2y \qquad \text{and} \qquad \vartheta^- = 0\,, \tag{36}$$

and

$$\sigma_0 = \cos x + \frac{1}{2}\cos 2y\,. \tag{37}$$

If we choose

$$\breve{q}_0 = \sin x \sin 2y\,, \tag{38}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\breve{q}_0^2}{2} - \tfrac{1}{h}\vartheta^+\breve{q}_0\mathrm{d}S = \coth 1 - \frac{1}{2}\coth 2 - \frac{3}{4}\,, \tag{39}$$

after integrating over one period in both directions. Hence enstrophy in the form of M (15) is not conserved in this simplest model, and we conclude that, with non-zero β, enstrophy is not conserved in approximation B.

3 Approximation C: enstrophy nonconservation

Enstrophy

To obtain an enstrophy conservation for approximation C, we multiply the modal equations M (57) by $\breve{q}_n$, integrate over the surface, and sum on n, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\mathrm{N}}\int\frac{\breve{q}_n^2}{2}\,\mathrm{d}S - \beta\sum_{n=0}^{\mathrm{N}}\int\triangle_n\breve{\sigma}_n\partial_x\breve{\psi}_n\mathrm{d}S = 0\,. \tag{40}$$

With $\beta = 0$ the enstrophy

$$\sum_{n=0}^{\mathrm{N}}\int\frac{\breve{q}_n^2}{2}\,, \tag{41}$$

is conserved. For non-zero β we form an equation for $\triangle_n\breve{\sigma}_n$ by combining the boundary conditions

$$\partial_t\triangle_n\breve{\sigma}_n - \tfrac{1}{h}\sum_{m=0}^{\mathrm{N}}\mathsf{p}_n^+\mathsf{p}_m^+\mathsf{J}\left(\breve{\psi}_m, \vartheta^+\right) - \mathsf{p}_n^-\mathsf{p}_m^-\mathsf{J}\left(\breve{\psi}_m, \vartheta^-\right) = 0\,. \tag{42}$$

We then cross-multiply (42) with the modal equations, integrate over the surface, sum on n, and combine with (40) to eliminate β

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\mathrm{N}}\int\frac{\breve{q}_n^2}{2}+\breve{q}_n\triangle_n\breve{\sigma}_n\,\mathrm{d}S=-\sum_{m=0}^{\mathrm{N}}\sum_{n=0}^{\mathrm{N}}\sum_{s=0}^{\mathrm{N}}\int\Xi_{mns}\triangle_n\breve{\sigma}_n\mathsf{J}\left(\breve{\psi}_m\,,\breve{q}_s\right)$$
$$+\sum_{m=0}^{\mathrm{N}}\sum_{n=0}^{\mathrm{N}}\tfrac{1}{h}\int\breve{q}_n\mathsf{p}_n^+\mathsf{p}_m^+\mathsf{J}\left(\breve{\psi}_m,\vartheta^+\right)-\breve{q}_n\mathsf{p}_n^-\mathsf{p}_m^-\mathsf{J}\left(\breve{\psi}_m,\vartheta^-\right)\,. \tag{43}$$

The right-hand-side of (43) is only zero in very special cases.

3.0.4 The simplest model

Consider the most crude truncation (N = 0). Because $\Xi_{000}=1$ and $p_0=1$, the terms on the second and third lines of (43) cancel each other, ensuring conservation of enstrophy with non-zero β

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\mathrm{N}}\int\frac{\breve{q}_0^2}{2}-\tfrac{1}{h}\breve{q}_0\left(\vartheta^+-\vartheta^-\right)\,\mathrm{d}S=0\,. \tag{44}$$

This simplest model is a very special case in which the following buoyancy variance

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\mathrm{N}}\int\frac{(\vartheta^+-\vartheta^-)^2}{2}=0\,, \tag{45}$$

and the enstrophy

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\mathrm{N}}\int\frac{(\triangle\breve{\psi}_0)^2}{2}=0\,, \tag{46}$$

are conserved. The enstrophy conservation (44) follows directly from (45) and (46) and the inversion relationship

$$\breve{q}_0=\triangle\breve{\psi}_0+\tfrac{1}{h}(\vartheta^+-\vartheta^-)\,. \tag{47}$$

In general, however, one cannot construct invariants analogous to (45) and (46), and the system does not conserves enstrophy with non-zero β.

3.0.5 An example of enstrophy nonconservation

Consider the simplest model with interior shear (N = 1). With constant buoyancy frequency we obtain, after many cancellations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{1}\int\frac{\breve{q}_n^2}{2}+\breve{q}_n\triangle_n\breve{\sigma}_n\,\mathrm{d}S=-\frac{1}{h}\int\breve{q}_1\mathsf{J}\left(\breve{\psi}_1\,,\vartheta^+-\vartheta^-\right)\mathrm{d}S=$$
$$\int\left(\vartheta^+-\vartheta^-\right)\mathsf{J}\left(\breve{\psi}_1\,,(\triangle-\pi^2)\breve{\psi}_1\right)\mathrm{d}S+2\sqrt{2}\int\vartheta^+\mathsf{J}\left(\breve{\psi}_1,\vartheta^-\right)\mathrm{d}S\,, \tag{48}$$

where the last equality follows from using the inversion relationship.

We construct an example in which enstrophy is not conserved with non-zero β. For simplicity we choose $\breve{\psi}_1=\sin x$ so that the first integral on the second row of (48) vanishes identically. As for the surface fields, we choose

$$\vartheta^+=\cos x\,\cos y\quad\text{and}\quad\vartheta^-=\sin y\,. \tag{49}$$

All fields are periodic with same period. Integrating (48) over one period we obtain

$$\int \vartheta^{+} \mathsf{J}\left(\breve{\psi}_1, \vartheta^{-}\right) \mathrm{d}S = \frac{1}{4} \neq 0\,. \tag{50}$$

Thus enstrophy, in a form analogous to M (15), is not conserved in this simple example. We therefore conclude that enstrophy is not generally conserved in approximation C.